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Worst-case conditional value-at-risk and conditional expected shortfall based on covariance information

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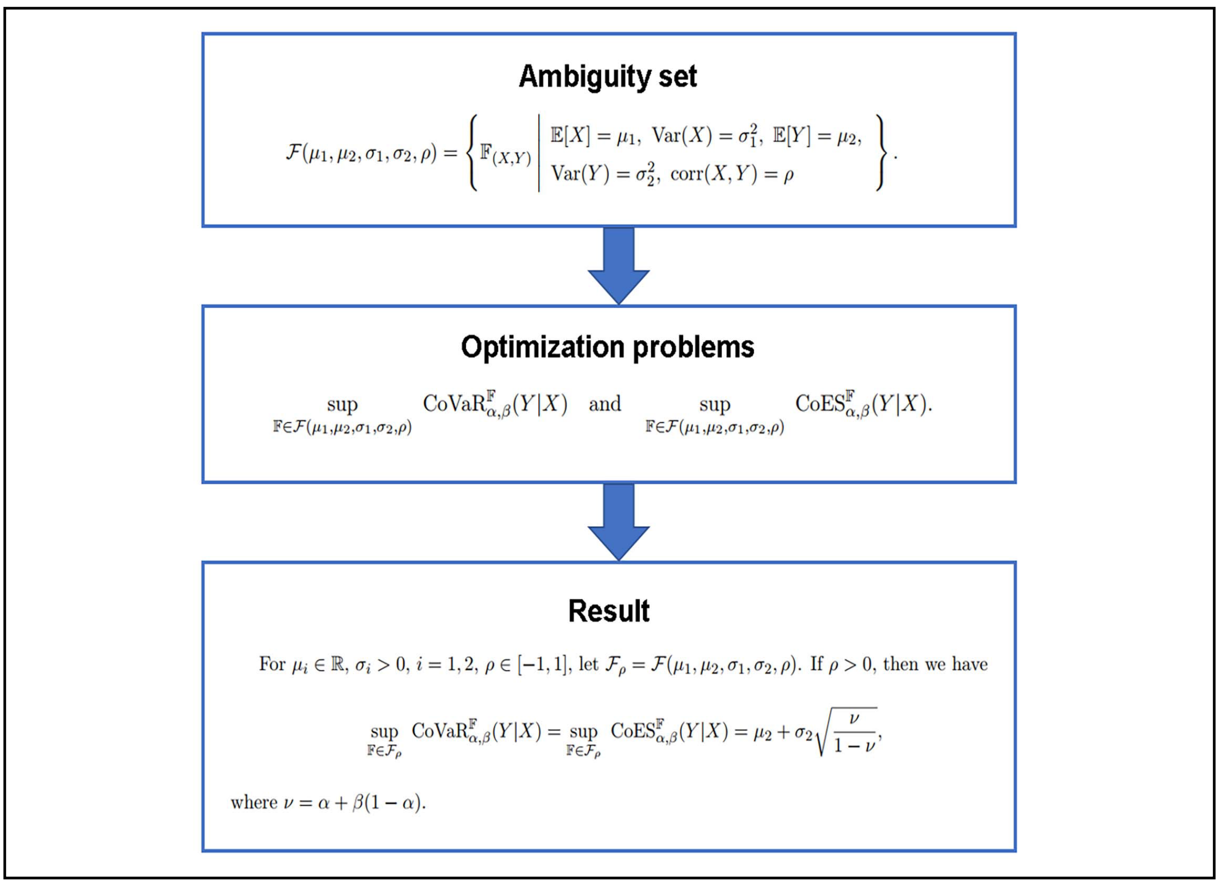
Graphical abstract



We construct the above new ambiguity set, then propose the optimization problem of CoVaR and CoES based on this ambiguity set, and give the theoretical results.

Public summary

- The relationship between CoVaR, CoES and dependence structure are investigated.
- In case where the first two marginal moments are known, the closed-form solution and the value of the worst-case CoVaR and CoES are derived.
- The worst-case CoVaR and CoES under mean and covariance information are investigated.

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Abstract: In this paper, we study the worst-case conditional value-at-risk (CoVaR) and conditional expected shortfall (CoES) in a situation where only partial information on the underlying probability distribution is available. In the case of the first two marginal moments are known, the closed-form solution and the value of the worst-case CoVaR and CoES are derived. The worst-case CoVaR and CoES under mean and covariance information are also investigated.

Keywords: conditional value-at-risk; conditional expected shortfall; distributional uncertainty

CLC number: O211.9 **Document code:** A

2020 Mathematics Subject Classification: 90C17; 91G70

1 Introduction

Modern risk management often requires the evaluation of risks under multiple scenarios. For instance, in the fundamental review of the trading book of Basel Ⅳ[1], banks need to evaluate the risk of their portfolios under stressed scenarios including the model generated from data during the 2007 financial crisis. In the aftermath of the financial crisis, there has been growing interest in measuring systemic risk, which refers to the risk that an event at the company level could trigger severe instability or collapse an entire industry or economy. Capital requirements are closely linked to an institution's contribution to the overall risk of the financial system and not merely to its individual risk.

Measuring the contribution of each institution to the overall systemic risk can help regulators inhibit the tendency to generate systemic risk by identifying institutions that make significant contributions to systemic risk. Starting with the seminal paper of Adrian and Brunnermeier[2] (first published online in 2008), many methods have been proposed for measuring systemic risk[3, 4]. While the conditional value-at-risk (CoVaR) proposed by Ref. [2] described the VaR of the financial system conditional on an institution being in financial distress, Girardi and Ergün[5] modified the computation of CoVaR in Ref. [2] by changing the definition of financial distress from the loss of an institution being exactly its VaR to being no less than its VaR. Specifically, for a set of financial institutions (or portfolio) $X=(X_1,\cdots,X_n)$, we denote by $S=X_1+\cdots+X_n$ as the total systemic risk. The value-at-risk (VaR) of an institution X_i at level α is defined as the α-quantile function of X_i, that is, $\mathrm{VaR}_\alpha(X_i)=\inf\{x\in\mathbb{R}:\mathbb{P}(X_i\leqslant x)\geqslant\alpha\}$. The notation $\mathrm{CoVaR}_{\alpha,\beta}(S|X_i)$ is defined as the VaR of the systemic risk S at level β, conditional on one of the institutions X_i beyond its VaR at level α, that is,

$$\mathrm{CoVaR}_{\alpha,\beta}(S|X_i)=\inf\{y\in\mathbb{R}:\mathbb{P}(S\leqslant y|X_i\geqslant\mathrm{VaR}_\alpha(X_i))>\beta\} \tag{1}$$

We use the right-continuous version of CoVaR in this study, that is, $>\beta$ instead of $\geqslant\beta$ in Eq. (1). In the case of the right-continuous version, the worst-case value is reachable. Both definitions have the same worst-case value, which has no effect on our study. Huang and Uryasev[6] linked the systemic risk contribution of an institution to the increase in the CoVaR of the entire financial system while the institution is under distress. Acharya et al.[4] proposed the marginal expected shortfall (ES) to measure the contributions of financial institutions to systemic risk, whose mathematical expression has been generalized to the following conditional expected shortfall (CoES)[2], at level β, conditional on one of the institutions X_i beyond its VaR at level α,

$$\mathrm{CoES}_{\alpha,\beta}(S|X_i):=\frac{1}{1-\beta}\int_\beta^1\mathrm{CoVaR}_{\alpha,s}(S|X_i)\,\mathrm{d}s \tag{2}$$

Notably, CoVaR and CoES, defined by Eqs. (1) and (2), and their transformers have played an essential role in measuring the system risk. Detailed discussions and their applications in economics, finance, and other fields can be found in Refs. [2, 5, 7, 8], as well as the references therein.

Measuring systemic risk requires the knowledge of its probability distribution. In most practices, the exact form of the distribution is often lacking, and only sample data are available for estimating the distribution, which is inevitably prone to sampling error. This situation, wherein the probability distribution of uncertain outcomes cannot be uniquely identified, is referred to as distributional uncertainty. The question of how to account for distributional uncertainty in decision-making has been of central interest in several fields, including economics, finance, control system, and operations research/ management science. One modeling paradigm that has been successfully adopted in all these fields to address this issue is distributionally robust optimization (DRO). In the standard form of DRO, we characterize one's (partial) information by specifying an uncertainty set $\mathcal{F}$, which is also known as an ambiguity set, instead of an underlying probability distribution that is known exactly. Various types of uncertainty sets $\mathcal{F}$ have been proposed in the literature. One common way of defining the set $\mathcal{F}$ is by specifying the moments of the distribution. The earlier works of Popescu[9], Bertsimas et al.[10], Delage and Ye[11], and Natarajan et al.[12] have considered the case where the uncertainty set is specified in terms of the first two moments. More recently, Wiesemann et al.[13] considered a case in which the uncertainty set was described through supports and higher-order moments. In this study, we consider the worst-case CoVaR and CoES under moment uncertainty. Specifically, we consider the following optimization problems:

$$\sup_{\mathbb{F}\in\mathcal{F}} \mathrm{CoVaR}^{\mathbb{F}}(S|X_i) \quad \text{and} \quad \sup_{\mathbb{F}\in\mathcal{F}} \mathrm{CoES}^{\mathbb{F}}(S|X_i) \tag{3}$$

where $\mathcal{F}$ is an uncertainty set specifying the mean vector and the covariance of X, and $\mathrm{CoVaR}^{\mathbb{F}}$ and $\mathrm{CoES}^{\mathbb{F}}$ represent the calculation for $\mathrm{CoVaR}(S|X_i)$ and $\mathrm{CoES}(S|X_i)$ under the joint distribution $\mathbb{F}$, respectively. While the current stduy focuses on moment-based uncertainty sets, we should point out here that the uncertainty set can also be defined according to a certain distance over distributions, such as KL divergence and Wasserstein metric.

2 Preliminaries

Let $(\Omega,\mathcal{B},\mathbb{P})$ be an atomless probability space, and $(\Omega,\mathcal{B},\mathbb{P})^n$ as its n-dimensional product space, where Ω is a set of possible states of nature and $\mathcal{B}$ is a σ-algebra on Ω. The random variable is a measurable real-valued functional on $(\Omega,\mathcal{B},\mathbb{P})$. For a random vector (random variable) $X=(X_1,\dots,X_n)$, its distribution function is defined by $\mathbb{F}(x_1,\dots,x_n)=\mathbb{P}(X_1\leqslant x_1,\dots, X_n\leqslant x_n)$ for $(x_1,\dots,x_n)\in\mathbb{R}^n$, and we denote the distribution function of the random vector (random variable) X by $\mathbb{F}_X$. The notation δ_x represents the point mass at $x\in\mathbb{R}^n$. The left and right quantile function of a univariate distribution F are denoted by F^{-1} and F^{-1+}, respectively. For a mapping $f:(\Omega,\mathcal{B},\mathbb{P})^n\to\mathbb{R}$, the notation $f^{\mathbb{F}}(X)$ indicates that it has the same value as $f(X)$, where X is a random vector (random variable) in $(\Omega,\mathcal{B},\mathbb{P})^n$ and $\mathbb{F}$ is its distribution. In this study, both notation, $f^{\mathbb{F}}(X)$ and $f(X)$, are used. For a random variable X, we denote the mean and the variance of X by $\mathbb{E}[X]$ and $\mathrm{Var}(X)$, respectively, and for a random vector X, we denote the mean vector and the covariance matrix by $\mathbb{E}[X]$ and $\mathrm{Cov}(X)$, respectively.

Notably, VaR and ES are two popular and important risk measures in financial practice. The left and right VaRs of a random variable X at level $\alpha\in(0,1)$ are defined by $\mathrm{VaR}_\alpha(X)=F_X^{-1}(\alpha)$ and $\mathrm{VaR}_\alpha^+(X)=F_X^{-1+}(\alpha)$, respectively. The ES of a random variable X at level $\alpha\in[0,1)$ is defined by $\mathrm{ES}_\alpha(X)=1/(1-\alpha)\int_\alpha^1\mathrm{VaR}_s(X)\,\mathrm{d}s$. Furthermore, CoVaR and CoES are defined by Eqs. (1) and (2) in the Introduction, respectively.

2.1 Worst-case systemic risk measures

In this study, we examine the worst-case CoVaR and CoES with an uncertainty set based on moment constraints. Specifically, for a portfolio $X=(X_1,\cdots,X_n)$, let $S=X_1+\cdots+X_n$, and we consider the following optimization problems:

$$\sup_{\mathbb{F}\in\mathcal{F}_{\mu,\Sigma}} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(S|X_i) \quad \text{and} \quad \sup_{\mathbb{F}\in\mathcal{F}_{\mu,\Sigma}} \mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(S|X_i) \tag{4}$$

where the uncertainty set $\mathcal{F}$ is defined by moment information, that is

$$\mathcal{F}_{\mu,\Sigma}=\{\mathbb{F}:\mathbb{E}^{\mathbb{F}}[X]=\mu,\ \mathrm{Cov}^{\mathbb{F}}(X)=\Sigma\},$$

where $\mu\in\mathbb{R}^n$, $\Sigma\in\mathbb{R}^{n\times n}$ is a given semi-positive matrix. Our aim is to investigate the optimization problems in (4).

The objective function of (4) only depends on X_i and S. The uncertainty constraint can be replaced by the following set:

$$\mathcal{F}_{i,\mu,\Sigma}=\{\mathbb{F}_{(X_i,S)}:\mathbb{E}^{\mathbb{F}}[X]=\mu,\ \mathrm{Cov}^{\mathbb{F}}(X)=\Sigma\},$$

where $\mathbb{F}_{(X_i,S)}$ is the joint distribution function of (X_i,S). Applying the general projection property in Ref. [9] (see also Ref. [14, Lemma 2.4]), $\mathcal{F}_{i,\mu,\Sigma}$ equals to the following uncertainty set:

$$\mathcal{F}=\Big\{\mathbb{F}\text{ is a cdf on }\mathbb{R}^2:\mathbb{E}^{\mathbb{F}}[(X,Y)]=\Big(\mu_i,\sum_{i=1}^n\mu_i\Big), \\ \mathrm{Cov}^{\mathbb{F}}(X,Y)=(e_i,\mathbf{1})^\top\Sigma(e_i,\mathbf{1})\Big\} \tag{5}$$

where $\mathbf{1}=(1,\dots,1)\in\mathbb{R}^n$, and $e_i=(0,\cdots,1,\cdots,0)\in\mathbb{R}^n$ is the vector whose ith element equals to 1 and the other elements are all zero. These arguments inspire us to consider the following general optimization problems:

$$\sup_{\mathbb{F}\in\mathcal{F}(\mu,\Sigma)} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X) \quad \text{and} \quad \sup_{\mathbb{F}\in\mathcal{F}(\mu,\Sigma)} \mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X) \tag{6}$$

where $\mathcal{F}(\mu,\Sigma)$ is a mean-variance uncertainty set of two-dimensional random vectors defined by

$$\mathcal{F}(\mu,\Sigma)=\{\mathbb{F}\text{ is a cdf on }\mathbb{R}^2:\mathbb{E}^{\mathbb{F}}[(X,Y)]=\mu,\ \mathrm{Cov}^{\mathbb{F}}(X,Y)=\Sigma\},$$

where $\mu\in\mathbb{R}^2$, and $\Sigma\in\mathbb{R}^{2\times2}$ is a semi-positive matrix. It is easy to verify that the original optimization problems in (4) are a special case of optimization problems in (6) with $\boldsymbol{\mu}=(\mu_i,\sum_{i=1}^n\mu_i)$ and $\Sigma=(e_i,\mathbf{1})^\top\Sigma_n(e_i,\mathbf{1})$, where Σ_n is the covariance of the random vector $(X_1,\dots,X_n)$.

In the remainder of this paper, we aim to solve the optimization problems in (6). We always assume that all considered random variables X in (6) satisfy that F_X is continuous at $\mathrm{VaR}_\alpha(X)$ so that $\mathbb{P}(X\leqslant\mathrm{VaR}_\alpha(X))=1-\mathbb{P}(X\geqslant\mathrm{VaR}_\alpha(X))=\alpha$. Some preliminaries on the copula theory are needed, and these will be introduced in the next subsection.

2.2 Copula

In this section, we recall the definition of two-dimensional copula that is a tool for separating dependence and marginal distributions. A two-dimensional copula is a function $C:[0,1]^2\to[0,1]$ that satisfies

(i) for any $u,v\in[0,1]$, $C(u,0)=0=C(0,v)$, $C(u,1)=u$ and $C(1,v)=v$;

(ii) for any $u_1,u_2,v_1,v_2\in[0,1]$ such that $u_1\leqslant u_2$ and $v_1\leqslant v_2$,

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$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1)\geqslant 0.$$

For a two-dimensional random vector (X,Y) with joint distribution $\mathbb{F}_{(X,Y)}$, its copula is denoted by C. By Sklar's theorem[15], the joint distribution can be expressed as

$$\mathbb{F}_{(X,Y)}(x,\ y)=C(F_X(x),\ F_Y(y)) \tag{7}$$

An extremal copula is the comonotonicity copula, defined as

$$C^M(u,v)=\min\{u,v\}.$$

For any copula C, it holds that $C\leqslant C^M$ pointwisely. We refer the readers to Ref. [15] for more details on copula.

By the relation between joint distribution function and the copula in Eq. (7), it is natural to denote thus:

$$\mathrm{CoVaR}^C_{\alpha,\beta}(G|F)=\mathrm{CoVaR}_{\alpha,\beta}(Y|X)$$

and

$$\mathrm{CoES}^C_{\alpha,\beta}(G|F)=\mathrm{CoES}_{\alpha,\beta}(Y|X),$$

where F and G are the marginal distributions of X and Y, respectively, and C is the copula of (X,Y). Hence, we have

$$\begin{aligned}&\mathrm{CoVaR}^C_{\alpha,\beta}(G|F)=\\ &\inf\{y\in\mathbb{R}:\mathbb{P}(Y\leqslant y|X\geqslant \mathrm{VaR}_\alpha(X))>\beta\}=\\ &\inf\left\{y\in\mathbb{R}:\frac{\mathbb{P}(Y\leqslant y,X\geqslant \mathrm{VaR}_\alpha(X))}{\mathbb{P}(X\geqslant \mathrm{VaR}_\alpha(X))}>\beta\right\}=\\ &\inf\left\{y\in\mathbb{R}:\frac{G(y)-C(\alpha,G(y))}{1-\alpha}>\beta\right\}\end{aligned} \tag{8}$$

The above-stated formula illustrates that the value of $\mathrm{CoVaR}^C_{\alpha,\beta}(G|F)$ does not depend on the marginal distribution F. The following proposition, which collects the results of Ref. [16, Theorem 3.4] shows that $C_1\leqslant C_2$ pointwisely implies $\mathrm{CoVaR}^{C_1}_{\alpha,\beta}\leqslant\mathrm{CoVaR}^{C_2}_{\alpha,\beta}$ and $\mathrm{CoES}^{C_1}_{\alpha,\beta}\leqslant\mathrm{CoES}^{C_2}_{\alpha,\beta}$ for all $\alpha,\beta\in(0,1)$, and it is useful throughout the paper.

Proposition 2.1. Let $\alpha,\beta\in(0,1)$, and let F_1, F_2 and G be three univariate distributions. For any copula C, we have

$$\mathrm{CoVaR}^C_{\alpha,\beta}(G|F_1)=\mathrm{CoVaR}^C_{\alpha,\beta}(G|F_2)$$

and

$$\mathrm{CoES}^C_{\alpha,\beta}(G|F_1)=\mathrm{CoES}^C_{\alpha,\beta}(G|F_2).$$

Moreover, if C_1,C_2 are two copulas such that $C_1\leqslant C_2$ pointwisely, then we have

$$\mathrm{CoVaR}^{C_1}_{\alpha,\beta}(G|F_1)\leqslant\mathrm{CoVaR}^{C_2}_{\alpha,\beta}(G|F_2)$$

and

$$\mathrm{CoES}^{C_1}_{\alpha,\beta}(G|F_1)\leqslant\mathrm{CoES}^{C_2}_{\alpha,\beta}(G|F_2).$$

Proof. By Eq. (8), we can immediately obtain $\mathrm{CoVaR}^C_{\alpha,\beta}(G|F_1)=\mathrm{CoVaR}^C_{\alpha,\beta}(G|F_2)$. Because CoES is formulated as the integral of CoVaR, we have $\mathrm{CoES}^C_{\alpha,\beta}(G|F_1)=\mathrm{CoES}^C_{\alpha,\beta}(G|F_2)$. To see the "Moreover" part, if $C_1\leqslant C_2$ pointwisely, then we have $C_1(\alpha,G(y))\leqslant C_2(\alpha,G(y))$ for all $y\in\mathbb{R}$. It follows from Eq. (8) that $\mathrm{CoVaR}^{C_1}_{\alpha,\beta}(G|F_1)\leqslant\mathrm{CoVaR}^{C_2}_{\alpha,\beta}(G|F_2)$. Noting that CoES is formulated as the integral of CoVaR, we obtain $\mathrm{CoES}^{C_1}_{\alpha,\beta}(G|F)\leqslant\mathrm{CoES}^{C_2}_{\alpha,\beta}(G|F)$. Hence, we complete the proof.

Since for any copula C, we have $C(u,v)\leqslant C^M$. The next proposition is a direct result followed by Proposition 2.1.

Proposition 2.2. Let C_0 be a copula function, and $\mathcal{C}=\{C|C_0\in\mathcal{C},C\leqslant C_0\}$. Consider two optimization problems

$$\text{(i)}\ \sup_{C\in\mathcal{C}}\mathrm{CoVaR}^C_{\alpha,\beta}(G|F)\quad\text{and}\quad\text{(ii)}\ \sup_{C\in\mathcal{C}}\mathrm{CoES}^C_{\alpha,\beta}(G|F).$$

Both maximization problems can both be attained at C_0. Specifically, if $C^M\in\mathcal{C}$, then the optimization problems (i) and (ii) can be attained at C^M, and the values of (i) and (ii) are $\mathrm{VaR}^+_\nu(Y)$ and $\mathrm{ES}_\nu(Y)$, respectively, where $\nu=\alpha+\beta(1-\alpha)$ and Y has distribution G.

Proof. The maximizer can be attained at C_0, immediately following Proposition 2.1. If $C^M\in\mathcal{C}$, and $C^M\geqslant C$ for all copulas C, the maximizer can be attained at C^M. Moreover, in this case, we obtain

$$\begin{aligned}&\sup_{C\in\mathcal{C}}\mathrm{CoVaR}^C_{\alpha,\beta}(G|F)=\\ &\inf\{y:\mathbb{P}(Y\leqslant y|X\geqslant\mathrm{VaR}_\alpha(X))>\beta,\\ &X\sim F,\ Y\sim G,\ (X,Y)\text{ has copula }C^M\}=\\ &\inf\left\{y:\frac{G(y)-\min\{G(y),\alpha\}}{1-\alpha}>\beta\right\}=\\ &G^{-1+}(\alpha+\beta(1-\alpha))=\mathrm{VaR}^+_\nu(Y),\end{aligned}$$

and

$$\begin{aligned}&\sup_{C\in\mathcal{C}}\mathrm{CoES}^C_{\alpha,\beta}(G|F)=\\ &\frac{1}{1-\beta}\int_\beta^1 G^{-1+}(\alpha+(1-\alpha)s)\,\mathrm{d}s=\\ &\frac{1}{1-\nu}\int_\nu^1 G^{-1+}(s)\,\mathrm{d}s=\mathrm{ES}_\nu(Y).\end{aligned}$$

Hence, we complete the proof.

Proposition 2.2 provides a natural idea to solve the optimization problem $\sup_{\mathbb{F}\in\mathcal{F}}R^{\mathbb{F}}(Y|X)$, where $R=\mathrm{CoVaR}_{\alpha,\beta}$ or $\mathrm{CoES}_{\alpha,\beta}$, in two steps. First, fix the marginal distributions as F and G, and calculate $R_{F,G}=\sup_{C\in\mathcal{C}_{F,G}}R^C(G|F)$ where $\mathcal{C}_{F,G}=\{C:C(F,G)\in\mathcal{F}\}$. Second, we calculate $\sup_{F,G}R_{F,G}$. If for all F,G, the set $\mathcal{C}_{F,G}=\{C:C(F,G)\in\mathcal{F}\}$ contains a copula $C_{F,G}$ such that $C_{F,G}\geqslant C$ pointwisely for all $C\in\mathcal{C}_{F,G}$, then it follows from Proposition 2.2 that $\sup_{\mathbb{F}\in\mathcal{F}}R^{\mathbb{F}}(Y|X)=\sup_{F,G}R^{C_{F,G}}(G|F)$.

3 Worst-case CoVaR and CoES under moment constraints

3.1 Marginal information

In this subsection, we consider the worst-case CoVaR and CoES in the case that the uncertainty set contains the information of the first two marginal moments, that is,

$$\sup_{\mathbb{F}\in\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2)}\mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X)\quad\text{and}\quad\sup_{\mathbb{F}\in\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2)}\mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X) \tag{9}$$

where $\mu_i\in\mathbb{R}$ and $\sigma_i>0$ for $i=1,2$, and

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$$\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2) = \{\mathbb{F}_{(X,Y)} : \mathbb{E}[X]=\mu_1,\ \mathrm{Var}(X)=\sigma_1^2,\ \mathbb{E}[Y]=\mu_2,\ \mathrm{Var}(Y)=\sigma_2^2\} \tag{10}$$

The following lemma, which plays an important role in the proof of the main theorems presented in this paper, is a direct result of Refs. [17, Theorem 1] and [14, Theorem 2.9].

Lemma 3.1. For $\mu\in\mathbb{R}$, $\sigma>0$ and $\alpha\in[0,1)$, it holds that

$$\sup_{\mathbb{E}[Y]=\mu,\mathrm{Var}(Y)=\sigma^2} \mathrm{VaR}_\alpha^+(Y) = \sup_{\mathbb{E}[Y]=\mu,\mathrm{Var}(Y)=\sigma^2} \mathrm{ES}_\alpha(Y) = \mu+\sigma\sqrt{\frac{\alpha}{1-\alpha}}.$$

The next result shows the value and the closed-form solution of the worst-case CoVaR and CoES defined by (9).

Theorem 3.1. For $\mu_i\in\mathbb{R}$, $\sigma_i>0$, $i=1,2$, and $\mathcal{F}=\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2)$ defined by (10), we have

$$\sup_{\mathbb{F}\in\mathcal{F}} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X) = \sup_{\mathbb{F}\in\mathcal{F}} \mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X) = \mu_2+\sigma_2\sqrt{\frac{\nu}{1-\nu}},$$

where $\nu=\alpha+\beta(1-\alpha)$, and the supremum can be attained at the joint distribution $(x,y)\mapsto\min\{F(x),G(y)\}$, where the mean and variance of F is μ_1 and σ_1^2, respectively, and G is a two-point distribution, defined as

$$G=\nu\delta_{\mu_2-\sigma_2\sqrt{\frac{1-\nu}{\nu}}}+(1-\nu)\delta_{\mu_2+\sigma_2\sqrt{\frac{\nu}{1-\nu}}}.$$

Proof. For any (X,Y), it follows from Proposition 2.2 that

$$\mathrm{CoVaR}_{\alpha,\beta}(Y|X)\leqslant\mathrm{VaR}_\nu^+(Y)\quad\text{and}\quad\mathrm{CoES}_{\alpha,\beta}(Y|X)\leqslant\mathrm{ES}_\nu(Y).$$

Therefore, we have

$$\sup_{\mathbb{F}\in\mathcal{F}} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X)\leqslant\sup_{\mathbb{E}[Y]=\mu_2,\mathrm{Var}(Y)=\sigma_2^2}\mathrm{VaR}_\nu^+(Y)=\mu_2+\sigma_2\sqrt{\frac{\nu}{1-\nu}},$$

and

$$\sup_{\mathbb{F}\in\mathcal{F}} \mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X)\leqslant\sup_{\mathbb{E}[Y]=\mu_2,\mathrm{Var}(Y)=\sigma_2^2}\mathrm{ES}_\nu(Y)=\mu_2+\sigma_2\sqrt{\frac{\nu}{1-\nu}},$$

where the equalities in the two formulas given above follow from Lemma 3.1. On the other hand, one can verify that the worst-case value of CoVaR and CoES can be attained at the distribution given in the theorem. Thus, we have

$$\sup_{\mathbb{F}\in\mathcal{F}} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X) = \sup_{\mathbb{F}\in\mathcal{F}} \mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X) = \mu_2+\sigma_2\sqrt{\frac{\nu}{1-\nu}}.$$

This completes the proof.

Theorem 3.1 illustrates that the closed-form solution of (9) does not depend on the form of the distribution of X but requires a comonotonicity copula structure of (X,Y). In other words, we can derive different forms of the closed-form solution by constructing different forms of the distribution of X. The following is a special case of the closed-form solution defined in Theorem 3.1.

Example 3.1. Define

$$F=\nu U\left(\mu_1-\sigma_1^*\sqrt{\frac{1-\nu}{\nu}}-\varepsilon,\ \mu_1-\sigma_1^*\sqrt{\frac{1-\nu}{\nu}}+\varepsilon\right)+(1-\nu)\delta_{\mu_1+\sigma_1^*\sqrt{\frac{\nu}{1-\nu}}},$$

and

$$G=\nu\delta_{\mu_2-\sigma_2\sqrt{\frac{1-\nu}{\nu}}}+(1-\nu)\delta_{\mu_2+\sigma_2\sqrt{\frac{\nu}{1-\nu}}},$$

where $\nu=\alpha+\beta(1-\alpha)$, $\varepsilon,\sigma_1^*\geqslant0$ satisfy $(\nu\varepsilon^2)/3+\sigma_1^{*2}=\sigma_1^2$, and $U(a,b)$ represents a uniform distribution on $[a,b]$. It is easy to verify that the mean and variance of F are μ_1 and σ_1^2; hence, $\min\{F,G\}$ is a closed-form solution of (9). Moreover, suppose that $\varepsilon^2\leqslant3\sigma_1^2/(\nu(4-3\nu))$. Then, we have

$$\mu_1-\sigma_1^*\sqrt{\frac{1-\nu}{\nu}}+\varepsilon\leqslant\mu_1+\sigma_1^*\sqrt{\frac{\nu}{1-\nu}},$$

and the correlation coefficient of (X,Y) with joint distribution $\min\{F,G\}$ is

$$\mathrm{corr}(X,Y)=\sqrt{1-\frac{\nu\varepsilon^2}{3\sigma_1^2}}\in\left[\frac{3(1-\nu)}{1+3(1-\nu)},1\right].$$

In particular, if $\varepsilon=0$, then F reduces to a two-point distribution. In this case, the correlation coefficient equals to 1.

3.2 Mean-covariance information

In this subsection, we consider the optimization problems in (6) when the uncertainty set $\mathcal{F}$ contains the information of the mean vector and covariance with a fixed correlation coefficient, that is,

$$\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)=\left\{\mathbb{F}_{(X,Y)}\ \middle|\ \begin{array}{l}\mathbb{E}[X]=\mu_1,\ \mathrm{Var}(X)=\sigma_1^2,\ \mathbb{E}[Y]=\mu_2,\\ \mathrm{Var}(Y)=\sigma_2^2,\ \mathrm{corr}(X,Y)=\rho\end{array}\right\} \tag{11}$$

where $\mu_i\in\mathbb{R}$, $\sigma_i>0$ for $i=1,2$, $\rho\in[-1,1]$ and $\mathrm{corr}(X,Y)$ represents the correlation coefficient of (X,Y). It is noteworthy that $\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)\subseteq\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2)$ for all $\rho\in[-1,1]$, where $\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2)$ is defined by Eq. (10). By Theorem 3.1, we have

$$\begin{aligned}&\sup_{\mathbb{F}\in\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)}\mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X)\leqslant\\&\sup_{\mathbb{F}\in\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)}\mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X)\leqslant\\&\mu_2+\sigma_2\sqrt{\frac{\nu}{1-\nu}}\end{aligned} \tag{12}$$

where $\nu=\alpha+\beta(1-\alpha)$. Recall Example 3.1, we know that the equalities of (12) hold if $\rho\geqslant3(1-\nu)/(1+3(1-\nu))$. A natural question is that what is the range of ρ that will make (12) become equalities? To answer this question, we first present the following lemma, which shows that the values of the worst-case CoVaR and CoES with the uncertainty set $\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)$ increase in ρ.

Lemma 3.2. For $\mu_i\in\mathbb{R}$, $\sigma_i>0$, $i=1,2$, $\rho\in[-1,1]$, let $\mathcal{F}_\rho:=\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)$ defined by Eq. (11). Both $\sup_{\mathbb{F}\in\mathcal{F}_\rho}\mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X)$ and $\sup_{\mathbb{F}\in\mathcal{F}_\rho}\mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X)$ increase in ρ.

Proof. We only give the proof of the case of CoVaR, as the case of CoES can be proved similarly. Let $-1\leqslant\rho_1<\rho_2\leqslant1$, and denote by $\theta=\sup_{\mathbb{F}\in\mathcal{F}_{\rho_1}}\mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X)$. We demonstrate that $\sup_{\mathbb{F}\in\mathcal{F}_{\rho_2}}\mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X)\geqslant\theta$. To see it, for any $\varepsilon>0$, there exists (X_1,Y_1) such that $\mathbb{F}_{(X_1,Y_1)}\in\mathcal{F}_{\rho_1}$ and $\mathrm{CoVaR}_{\alpha,\beta}(Y_1|X_1)\geqslant\theta-\varepsilon$. We denote the copula of (X_1,Y_1) by C_1. Let (X_1^c,Y_1^c) be a random vector, with the same marginal distributions as

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JUSTC, 2022, 52(5): 4

(X_1, Y_1) with the comonotonicity copula C^M and denote the correlation coefficient of (X_1^c, Y_1^c) by ρ_0. We consider the following two cases.

Case 1: If $\rho_1 < \rho_2 \leqslant \rho_0$, then denote by $\lambda = (\rho_0 - \rho_2)/(\rho_0 - \rho_1)$, and define (X_2, Y_2) that has the same marginal distributions as (X_1, Y_1) with the copula $C_2 = \lambda C_1 + (1-\lambda)C^M$. It can be observed that $\mathrm{corr}(X_2, Y_2) = \lambda\rho_1 + (1-\lambda)\rho_0 = \rho_2$; hence, $\mathbb{F}_{(X_2,Y_2)} \in \mathcal{F}_{\rho_2}$. As $C_1 \leqslant C_2$ pointwisely, it follows from Proposition 2.1 that

$$\theta - \varepsilon \leqslant \mathrm{CoVaR}_{\alpha,\beta}(Y_1|X_1) \leqslant \mathrm{CoVaR}_{\alpha,\beta}(Y_2|X_2) \leqslant \sup_{\mathbb{F}\in\mathcal{F}_{\rho_2}} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X).$$

Case 2: If $\rho_0 < \rho_2$, then denote by $X^c = \sigma_1(Y_1^c - \mu_2)/\sigma_2 + \mu_1$. We know that $\mathbb{E}[X^c] = \mu_1$, $\mathrm{Var}(X^c) = \sigma_1^2$ and $\mathrm{corr}(X^c, Y_1^c) = 1$. For $\lambda \in [0,1]$, let $X^c_{2,\lambda}$ with quantile function $F^{-1}_{X^c_{2,\lambda}} = \lambda F^{-1}_{X_1^c} + (1-\lambda)F^{-1}_{X^c}$, and $(X^c_{2,\lambda}, Y_1^c)$ has the comonotonicity copula C^M. One can calculate that $\mathbb{E}[X^c_{2,\lambda}] = \mu_1$,

$$\mathrm{Var}(X^c_{2,\lambda}) = \int_0^1 \left(\lambda F^{-1}_{X_1^c}(s) + (1-\lambda)F^{-1}_{X^c}(s)\right)^2 \mathrm{d}s - \mu_1^2 = (\lambda^2 + (1-\lambda)^2)\sigma_1^2 - (1 + \lambda^2 + (1-\lambda)^2)\mu_1^2 + 2\lambda(1-\lambda)\int_0^1 F^{-1}_{X_1^c}(s)F^{-1}_{X^c}(s)\,\mathrm{d}s,$$

and

$$\mathrm{corr}(X^c_{2,\lambda}, Y_1^c) = \frac{\int_0^1 \left(\lambda F^{-1}_{X_1^c}(s) + (1-\lambda)F^{-1}_{X^c}(s)\right)F^{-1}_{Y_1^c}(s)\,\mathrm{d}s - \mu_1\mu_2}{\sigma_2\sqrt{\mathrm{Var}(X^c_{2,\lambda})}} = \frac{\lambda\int_0^1 F^{-1}_{X_1^c}(s)F^{-1}_{Y_1^c}(s)\,\mathrm{d}s + (1-\lambda)\int_0^1 F^{-1}_{X^c}(s)F^{-1}_{Y_1^c}(s)\,\mathrm{d}s - \mu_1\mu_2}{\sigma_2\sqrt{\mathrm{Var}(X^c_{2,\lambda})}}.$$

We find that the function $\lambda \mapsto \mathrm{Var}(X^c_{2,\lambda})$ is continuous, and hence, the function $\lambda \mapsto f(\lambda) := \mathrm{corr}(X^c_{2,\lambda}, Y_1^c)$ is continuous. Also note that $f(0) = \mathrm{corr}(X^c, Y_1^c) = 1$ and $f(1) = \mathrm{corr}(X_1^c, Y_1^c) = \rho_0$, and $\rho_2 \in (\rho_0, 1]$. There exists $\lambda^* \in [0,1]$ such that $f(\lambda^*) = \mathrm{corr}(X^c_{2,\lambda^*}, Y_1^c) = \rho_2$. Now, let $a > 0$, $b \in \mathbb{R}$ and $X_2 = aX^c_{2,\lambda^*} + b$ such that $\mathbb{E}[X_2] = \mu_1$ and $\mathrm{Var}(X_2) = \sigma_1^2$. It can be seen that $\mathrm{corr}(X_2, Y_1^c) = \mathrm{corr}(X^c_{2,\lambda^*}, Y_1^c) = \rho_2$, and (X_2, Y_1^c) has the comonotonicity copula C^M; hence, $\mathbb{F}_{(X_2,Y_1^c)} \in \mathcal{F}_{\rho_2}$. From Proposition 2.1, we have

$$\theta - \varepsilon \leqslant \mathrm{CoVaR}_{\alpha,\beta}(Y_1|X_1) \leqslant \mathrm{CoVaR}_{\alpha,\beta}(Y_1^c|X_1^c) = \mathrm{CoVaR}_{\alpha,\beta}(Y_1^c|X_2) \leqslant \sup_{\mathbb{F}\in\mathcal{F}_{\rho_2}} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X),$$

where the second inequality holds because (X_1, Y_1) and (X_1^c, Y_1^c) have the same marginal distributions, and (X_1^c, Y_1^c) has the comonotonicity copula C^M.

Combing Cases 1 and 2, and noting that $\varepsilon > 0$ is arbitrary, we have $\sup_{\mathbb{F}\in\mathcal{F}_{\rho_2}} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X) \geqslant \theta$. This completes the proof.

Based on Lemma 3.2, if $\rho_0 \in [0,1]$ makes (12) become equalities, then so does for all $\rho \geqslant \rho_0$. The following theorem shows that zero is a lower bound of ρ that (12) holds as equalities.

Theorem 3.2. For $\mu_i \in \mathbb{R}$, $\sigma_i > 0$, $i = 1,2$, $\rho \in [-1,1]$, let $\mathcal{F}_\rho = \mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)$ defined by Eq. (11). If $\rho > 0$, then we have

$$\sup_{\mathbb{F}\in\mathcal{F}_\rho} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X) = \sup_{\mathbb{F}\in\mathcal{F}_\rho} \mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X) = \mu_2 + \sigma_2\sqrt{\frac{\nu}{1-\nu}},$$

where $\nu = \alpha + \beta(1-\alpha)$.

Proof. Let Y^* be a random variable with distribution

$$G = \nu\delta_{\mu_2 - \sigma_2\sqrt{\frac{1-\nu}{\nu}}} + (1-\nu)\delta_{\mu_2 + \sigma_2\sqrt{\frac{\nu}{1-\nu}}},$$

and we define $\mathcal{X} = \{X : \mathbb{E}[X] = \mu_1, \mathrm{Var}(X) = \sigma_1^2\}$. From Theorem 3.1, for any $X \in \mathcal{X}$ such that (X, Y^*) has the comonotonicity copula, we have $\mathrm{CoVaR}_{\alpha,\beta}(Y^*|X) = \mu_2 + \sigma_2\sqrt{\nu/(1-\nu)}$. We consider the following optimization problem:

$$\inf\{\mathrm{corr}(X, Y^*) : X \in \mathcal{X},\ (X, Y^*) \text{ has the comonotonicity copula}\} = \inf_{X\in\mathcal{X}} \frac{\int_0^1 \mathrm{VaR}_s(X)\mathrm{VaR}_s(Y^*)\,\mathrm{d}s - \mu_1\mu_2}{\sigma_1\sigma_2} = \inf_{X\in\mathcal{X}} \frac{a\mathrm{ES}^-_\nu(X) + b\mathrm{ES}_\nu(X) - \mu_1\mu_2}{\sigma_1\sigma_2} \tag{13}$$

where

$$a = \mu_2\nu - \sigma_2\sqrt{\nu(1-\nu)},$$
$$b = \mu_2(1-\nu) + \sigma_2\sqrt{\nu(1-\nu)},$$
$$\mathrm{ES}^-_\nu(X) = \frac{1}{\nu}\int_0^\nu \mathrm{VaR}_s(X)\,\mathrm{d}s.$$

We note that $\nu\mathrm{ES}^-_\nu(X) + (1-\nu)\mathrm{ES}_\nu(X) = \mu_1$. The problem (13) can be changed into

$$\inf_{X\in\mathcal{X}} \frac{\left(b - \frac{(1-\nu)a}{\nu}\right)\mathrm{ES}_\nu(X) + \frac{\mu_1 a}{\nu} - \mu_1\mu_2}{\sigma_1\sigma_2} = \frac{\left(b - \frac{(1-\nu)a}{\nu}\right)\inf_{X\in\mathcal{X}}\mathrm{ES}_\nu(X) + \frac{\mu_1 a}{\nu} - \mu_1\mu_2}{\sigma_1\sigma_2} = \frac{\left(b - \frac{(1-\nu)a}{\nu}\right)\mu_1 + \frac{\mu_1 a}{\nu} - \mu_1\mu_2}{\sigma_1\sigma_2} = 0,$$

where the first equality holds because $b - (1-\nu)a/\nu = \sigma_2\sqrt{(1-\nu)/\nu} \geqslant 0$, and the second equality follows from Ref. [18, Corollary 5]. Hence, for any $\rho \in (0,1]$, there exists $X_\varepsilon \in \mathcal{X}$ such that (X_ε, Y^*) has the comonotonicity copula with $\mathrm{corr}(X_\varepsilon, Y^*) \in (0, \rho)$, and $\mathrm{CoVaR}_{\alpha\beta}(Y^*|X_\varepsilon) = \mu_2 + \sigma_2\sqrt{\nu/(1-\nu)}$. Therefore,

$$\mu_2 + \sigma_2\sqrt{\frac{\nu}{1-\nu}} = \sup_{\mathbb{F}\in\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2)} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X) \geqslant \sup_{\mathbb{F}\in\mathcal{F}_\rho} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X) \geqslant \sup_{\mathbb{F}\in\mathcal{F}_{\mathrm{corr}(X_\varepsilon,Y^*)}} \mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X) \geqslant \mathrm{CoVaR}_{\alpha,\beta}(Y^*|X_\varepsilon) = \mu_2 + \sigma_2\sqrt{\frac{\nu}{1-\nu}},$$

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JUSTC, 2022, 52(5): 4

where $\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2)$ is defined by Eq. (10), and the second inequality follows from Lemma 3.2. Hence, we verify that $\sup_{\mathbb{F}\in\mathcal{F}_\rho}\mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X)=\mu_2+\sigma_2\sqrt{\nu/(1-\nu)}$ for all $\rho\in(0,1]$. Because CoES $\geqslant$ CoVaR, we have for any $\rho\in(0,1]$,

$$\begin{aligned}\mu_2+\sigma_2\sqrt{\frac{\nu}{1-\nu}}=&\sup_{\mathbb{F}\in\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2)}\mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X)\geqslant\\&\sup_{\mathbb{F}\in\mathcal{F}_\rho}\mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X)\geqslant\\&\sup_{\mathbb{F}\in\mathcal{F}_\rho}\mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X)=\\&\mu_2+\sigma_2\sqrt{\frac{\nu}{1-\nu}}.\end{aligned}$$

Hence, we have $\sup_{\mathbb{F}\in\mathcal{F}_\rho}\mathrm{CoES}^{\mathbb{F}}_{\alpha,\beta}(Y|X)=\mu_2+\sigma_2\sqrt{\nu/(1-\nu)}$ for all $\rho\in(0,1]$. This completes the proof.

Remark 3.1. Theorem 3.2 shows that it holds equalities for (12) if the underlying random vector (X,Y) is positively correlated.

Nevertheless, this result cannot hold for all $\rho\in[-1,1]$. For instance, if $\mathrm{corr}(X,Y)=-1$, then for any $\mathbb{F}_{(X,Y)}\in\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,-1)$, it holds that $X=\sigma_1(\mu_2-Y)/\sigma_2+\mu_1$. Let $\mathcal{A}=\{(X,Y):\mathbb{E}[Y]=\mu_2,\mathrm{Var}(Y)=\sigma_2^2,X=\sigma_1(\mu_2-Y)/\sigma_2+\mu_1\}$, and we have

$$\begin{aligned}&\sup_{\mathbb{F}\in\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,-1)}\mathrm{CoVaR}^{\mathbb{F}}_{\alpha,\beta}(Y|X)=\\&\sup_{(X,Y)\in\mathcal{A}}\inf\{y\in\mathbb{R}:\mathbb{P}(Y\leqslant y|X\geqslant\mathrm{VaR}_\alpha(X))>\beta\}=\\&\sup_{\mathbb{E}[Y]=\mu_2,\mathrm{Var}(Y)=\sigma_2^2}\inf\left\{y\in\mathbb{R}:\frac{\min\{1-\alpha,G(y)\}}{1-\alpha}>\beta\right\}=\\&\sup_{\mathbb{E}[Y]=\mu_2,\mathrm{Var}(Y)=\sigma_2^2}\mathrm{VaR}^+_{\beta(1-\alpha)}(Y)=\\&\mu_2+\sigma_2\sqrt{\frac{\beta(1-\alpha)}{1-\beta(1-\alpha)}}<\mu_2+\sigma_2\sqrt{\frac{\nu}{1-\nu}},\end{aligned}$$

where the fourth equality follows from Lemma 3.1. It is beyond current technology to calculate the value of the worst-case CoVaR and CoES under the uncertainty set $\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)$ when $\rho\leqslant0$. However, this could be an open question.

4 Simulation

In this section, we investigate the (worst-case) CoVaR and CoES in two cases: ① the true underlying distribution is a bivariate Gaussian or bivariate t-distribution with different correlation coefficients; ② the true underlying marginal distribution is a Pareto distribution with different copulas. In both cases, we include the worst-case CoVaR and CoES, where the partial information is induced by the given true underlying distribution.

4.1 CoVaR and CoES with Gaussian or *t*-distribution

In this subsection, we investigate the value of $\mathrm{CoVaR}_{\alpha,\beta}$ and $\mathrm{CoES}_{\alpha,\beta}$ when the true underlying distribution is a bivariate Gaussian or bivariate t-distribution. The marginal mean and variance of the Gaussian and t-distribution are chosen as $\mu_1=\mu_2=0$, $\sigma_1^2=\sigma_2^2=3$. In both cases, we consider three correlation coefficients, that is $\rho=0.2$, 0.5 and 0.9, and fix the parameter $\alpha=0.9$, and β ranges from 0.5 to 1. The worst-case $\mathrm{CoVaR}_{\alpha,\beta}$ and $\mathrm{CoES}_{\alpha,\beta}$ are derived by the moment information of the underlying distribution, the value of which is

$$\mu_2+\sigma_2\sqrt{\frac{\alpha+\beta(1-\alpha)}{1-\alpha-\beta(1-\alpha)}}=\sqrt{\frac{3(9+\beta)}{1-\beta}}\tag{14}$$

The corresponding values of $\mathrm{CoVaR}_{\alpha,\beta}$ and $\mathrm{CoES}_{\alpha,\beta}$ are showed in Figs. 1 and 2. Note that all given correlation coefficients are positive. From Theorem 3.2, the values of the worst-case CoVaR and CoES are also equal to Eq. (14) if the partial information is derived from the first two marginal moments and the correlation coefficient of the given distribution.

In Figs. 1 and 2, the values of CoVaR and CoES increase when the correlation coefficient ρ increases, and the worst-case CoVaR (CoES) is always larger than the CoVaR (CoES) generated by the true underlying distribution. Moreover, we find that the value of CoVaR (CoES) with a t-distribution is larger than that with Gaussian distribution when β is close to 1. This is because the t-distribution has a heavier tail than the Gaussian distribution does.

4.2 CoVaR and CoES with different copulas

In this subsection, we consider the values of CoVaR and CoES when the marginal distribution of the true underlying distribution is fixed as a Pareto distribution with an essential infimum $x_{\min}=1$ and tail parameter $k=3$, that is, $F(x)=(1-x^{-3})\mathbb{1}_{\{x\geqslant1\}}$, and copula changes. In all cases, the parameter $\alpha=0.9$, and β ranges from 0.5 to 1. In the following, we collect a few common copulas other than the comonotonicity copula, which is introduced in Section 2.2.

- Countermonotonicity copula: $C^C(u,v)=\max\{u+v-1,0\}$.
- Clayton copula: $C_\delta(u,v)=(\max\{u^{-\delta}+v^{-\delta}-1,0\})^{-\frac{1}{\delta}}$ for $\delta\in[-1,0)\cup(0,\infty)$.
- Independence copula: $C^I(u,v)=uv$.
- Gumbel copula: $G_\theta(u,v)=\mathrm{e}^{-((-\log u)^\theta+(-\log v)^\theta)^{\frac{1}{\theta}}}$ for $\theta>0$.

For Clayton and Gumbel copula, we let the parameter $\delta=-0.5$ and $\theta=2$, respectively. The worst-case $\mathrm{CoVaR}_{\alpha,\beta}$ and $\mathrm{CoES}_{\alpha,\beta}$ are derived from the moment information of the Pareto marginal distribution, that is

$$\mu+\sigma\sqrt{\frac{\alpha+\beta(1-\alpha)}{1-\alpha-\beta(1-\alpha)}}=\frac{3}{2}+\sqrt{\frac{3(9+\beta)}{4(1-\beta)}}\tag{15}$$

where $\mu=3/2$ and $\sigma^2=3/4$ denote the mean and variance of the marginal Pareto distribution, respectively. Fig. 3 shows the values of the worst-case CoVaR and CoES, and the CoVaR and CoES with Pareto marginal and different copulas introduced above.

As shown by Fig. 3, sorted the value of CoVaR of these copulas from small to large, we have countermonotonicity copula, Clayton copula ($\delta=-0.5$), independence copula, Gumbel copula ($\theta=2$) and comonotonicity copula, while the worst-case CoVaR is the largest. The case of CoES has a same performance. This is because $C^C\leqslant C_{-0.5}\leqslant C^I\leqslant G_2\leqslant C^M$ on $[0,1]\times[0,1]$. Hence, Proposition 2.1 can be applied. Note that the correlated coefficients of Gumbel and comonoton-

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JUSTC, 2022, 52(5): 4

icity copula are positive. By Theorem 3.2, in the cases of these two copulas, the value of the worst-case CoVaR and CoES are also equal to (15) if the partial information is derived from the first two moments of the marginal distribution and the correlation coefficient of the given copula.

5 Conclusions

In this paper, we study the worst-case $\mathrm{CoVaR}_{\alpha,\beta}$ and $\mathrm{CoES}_{\alpha,\beta}$ in case of model uncertainty with a known mean and covariance of the portfolio (X,Y), that is, the uncertainty set is $\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)$ defined by (11). When the correlation coefficient $\rho>0$, the values of the worst-case $\mathrm{CoVaR}_{\alpha,\beta}$ and $\mathrm{CoES}_{\alpha,\beta}$ are equal to a constant $\mu_2+\sigma_2\sqrt{\nu/(1-\nu)}$. To calculate the value of the worst-case CoVaR and CoES with uncertainty set $\mathcal{F}(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)$ when $\rho\leqslant 0$ is beyond current technology, and it could be an open question.

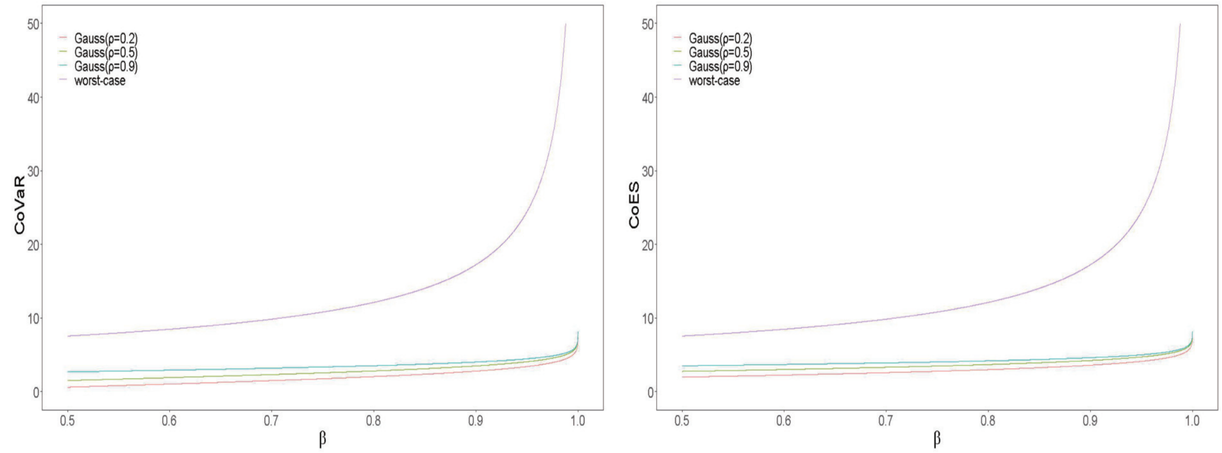

Fig. 1. CoVaR and CoES with bivariate Gaussian distributions.

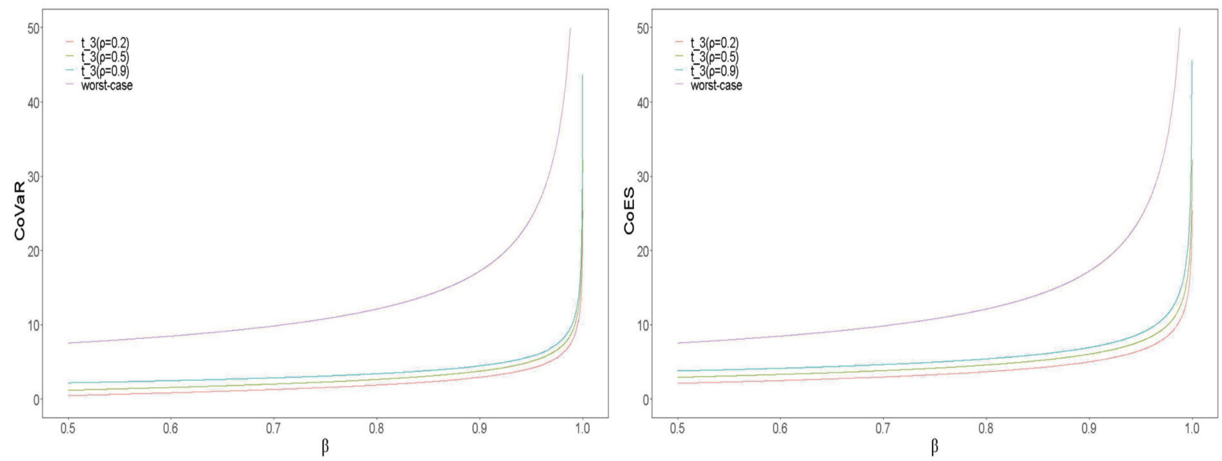

Fig. 2. CoVaR and CoES with bivariate *t*-distributions.

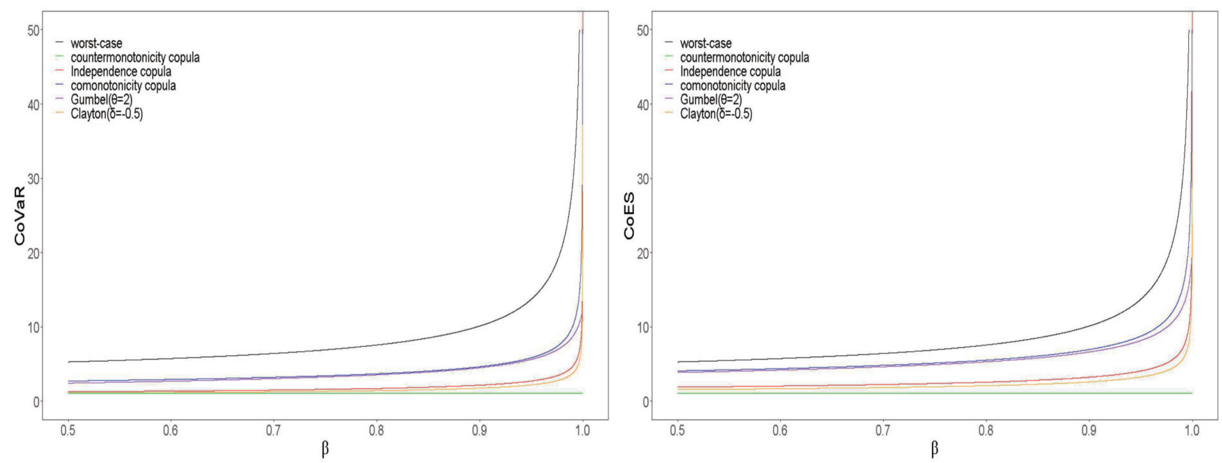

Fig. 3. CoVaR and CoES with Pareto marginal and different copulas.

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JUSTC, **2022**, 52(5): 4

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Conflict of interest

The authors declare that they have no conflict of interest.

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