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# Log-concavity of compound Poisson distributions

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**Abstract**: Log-concavity and total positivity of order 2 ($TP_2$) properties of two-parameter compound Poisson distributions $Q(x \mid \theta, \nu)$ with respect to $x$, $\theta$ and $\nu$ was studied by exploiting the interrelationships between log-concavity, $TP_2$ and reproductive property developed by the existing literatures. One application was also presented.

**Key words**: reproductive property; total positivity of order 2; reverse rule of order 2; increasing likelihood ratio; increasing failure rate; decreasing reversed hazard rate

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# 复合泊松分布的对数凹性质

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**摘要**:根据前人文献给出的对数凹性质、$TP_2$ 性质和再生性的关系,得到了双参数复合泊松分布关于其参数的对数凹性质以及 $TP_2$ 性质的相关结论.

**关键词**:再生性;二阶全正性;二阶反向正则;似然比递增;失效率递增;反向失效率递减

## 0 Introduction

Log-concave functions have many nice analytical properties, and play an important role in statistics, probability, economics, and other fields[1-3]. A nonnegative function $h:\mathbb{R}^n \to \mathbb{R}_+ \equiv [0, \infty)$ is said to be log-concave if, for all $x, y \in \mathbb{R}^n$ and for all $\alpha \in (0,1)$, we have

$$h(\alpha x + (1-\alpha)y) \geqslant [h(x)]^{\alpha}[h(y)]^{1-\alpha}.$$

If $h(x) > 0$ for all $x \in \mathbb{R}^n$, then an equivalent condition is

$$\ln h(\alpha x + (1-\alpha)y) \geqslant \alpha \ln h(x) + (1-\alpha)\ln h(y).$$

Total positivity of order 2 ($TP_2$) is a concept closely connected with log-concavity as shown in Lemma 1.1. Let $\mathbb{X}$ and $\mathbb{Y}$ be two subsets of the real line $\mathbb{R}$. A nonnegative function $\psi: \mathbb{X}\times\mathbb{Y} \to \mathbb{R}_+$ is said to be $TP_2$ if

$$\psi(x,y)\psi(x^*, y^*) \geqslant \psi(x^*, y)\psi(x, y^*) \tag{1}$$

whenever $x, x^* \in \mathbb{X}$, $y, y^* \in \mathbb{Y}$, and $x < x^*$ and $y < y^*$. If the inequality in (1) is reversed,

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then $\psi$ is said to be the reverse rule of order 2 ( $RR_2$ ). For more details on $TP_2$ and $RR_2$, see Ref. [3].

A variety of log-concavity results for families of distribution functions $F(x \mid \theta, \nu)$ and related functions in $x$ , $\theta$ or $\nu$ have been studied. Finner and Roters[4-5] and Das Gupta and Sarkar[6] studied the interrelationships between log-concavity, $TP_2$ and reproductive property. In view of such an interrelationship, Finner and Roters[5] obtained a series of log-concavity results not only for central but also for noncentral chi-square and F as well as for beta distributions.

The purpose of this short note is to investigate log-concavity and $TP_2$ ( $RR_2$ ) properties of two-parameter compound poisson distributions $Q(x \mid \theta,\nu)$ with respect to $x$ , $\theta$ and $\nu$ by similar arguments to those in Ref. [5]. The main results are given in Section 2. Section 1 gives the definition of reproductive property, and provides the interrelationships between log-concavity, $TP_2$ and reproductive property. A sufficient condition under which a compound Poisson distribution function possesses log-concavity is also recalled in Section 1. One application is presented in Section 3.

## 1 Preliminaries

First, recall the reproductivity given in Refs. [4-5]. Let ($\mathbb{X}$ , $\mathscr{A}$, $\mu$) denote a measure space which is in general assumed to be equal to ($\mathbb{R}$ , $\mathscr{B}$, $\lambda$) or ($\mathbb{Z}$ , $\mathscr{P}(\mathbb{Z})$, $\kappa$) , where $\lambda$ denotes the Lebesgue measure on the Borel $\sigma$-field $\mathscr{B}$ of the set of real numbers $\mathbb{R}$ , and $\kappa$ denotes the counting measure on the power set $\mathscr{P}(\mathbb{Z})$ of the set of integers $\mathbb{Z}$. Define $\mathbb{N}=\{1, 2, \cdots\}$ and $\mathbb{N}_0=\mathbb{N} \cup \{0\}$ , and let $\Theta$, $T \in \{(0,\infty), [0, \infty), \mathbb{N}, \mathbb{N}_0\}$ and $g: \mathbb{X}\times\Theta\times T \to \mathbb{R}_+$ be measurable in the first component. The function $g(x \mid \theta,\tau)$ is said to have the reproductive property in $\theta \in \Theta$ , denoted by RP ($\theta$) , if for every $\eta \in \Theta$ , there exists a probability measure $P_\eta$ on ($\mathbb{X}$, $\mathscr{A}$) with $P_\eta(\mathbb{R}_+ \cap \mathbb{X})=1$ such that for all $\theta \in \Theta$ ,

$$\int_{\mathbb{X}} g(x-y \mid \theta,\tau)\mathrm{d} P_\eta(y)= g(x \mid \theta+\eta,\tau), \mu\text{-a. s.}.$$

The first lemma below, which can be found in Refs. [4-5] and also in Ref. [6] with minor modification, reveals the relationship between $TP_2$ property and log-concavity.

**Lemma 1. 1** Let $f(x \mid \theta, \tau)$ be a density function defined on $\mathbb{X}\times\Theta\times T$ , and let $F(x \mid \theta, \tau)$ denote the corresponding distribution function. Suppose that $g(x \mid \theta,\tau) \in \{F(x \mid \theta, \tau), \overline{F}(x \mid \theta, \tau), f(x \mid \theta, \tau)\}$ is Borel measurable in $x \in \mathbb{X}$ and has RP ($\theta$) .

(a) If $g(x \mid \theta, \tau)$ is log-concave in $x$ for all $\theta$ and some $\tau$ , then $g(x \mid \theta, \tau)$ is $TP_2$ in $(x,\theta) \in \mathbb{X}\times\theta$ .

(b) If $g(x \mid \theta, \tau)$ is $TP_2$ in $(x,\theta) \in \mathbb{X}\times\Theta$ for some $\tau$ , then $g(x \mid \theta, \tau)$ is log-concave in $\theta$ for all $x$ .

**Lemma 1. 2**[6] Let $g(x \mid \theta, \tau)$ be as defined in Lemma 1. 1 with RP ($\theta$) property. If $g(x \mid \theta, \tau)$ is $TP_2$ in $(x,\tau) \in \mathbb{X}\times T$ for each $\theta$ , then $g(x \mid \theta, \tau)$ is $RR_2$ in $(\theta,\tau) \in \Theta\times T$ for each $x$ .

The third lemma, due to Ref. [8, Theorem 3. 4], states that the compound Poisson distribution function is log-concave if the underlying distribution possesses a decreasing density function on $\mathbb{R}_+$.

**Lemma 1. 3** Let $\{X_i, i \geqslant 1\}$ be a sequence of i. i. d. nonnegative random variables, and $N$ be a Poisson random variable independent of $\{X_i, i \geqslant 1\}$ . If $X_1$ has a decreasing density function on $\mathbb{R}_+$ , then the distribution function of $S_N=\sum_{i=1}^{N} X_i$ is log-concave.

For a distribution function $G$ , denote by $G^{k*}$ the $k$-fold convolution of $G$ , $k \in \mathbb{N}$ , and by $G^{0*}$ the distribution function of a degenerate random variable $X\equiv 0$. If $G$ has a density function $g$ , then denote by $g^{k*}$ the density function of $G^{k*}$ , where $k \in \mathbb{N}$. The next lemma is an immediate consequence of Ref. [8, Theorems 1. C. 11 and 1. C. 12]. Recall that a random variable $X$ or its distribution is said to be of increasing likelihood ratio (ILR)[resp. decreasing reversed hazard ratio (DRHR), increasing failure rate (IFR)] if $X$ has a log-concave density or mass function [resp. distribution function, survival function].

**Lemma 1.4** Let $G$ be a distribution function of a nonnegative random variable $X$. Then

(a) $g^{k*}(x)$ is $TP_2$ in $(k,x)\in\mathbb{N}\times\mathbb{R}$ if $X$ is ILR;

(b) $G^{k*}(x)$ is $TP_2$ in $(k,x)\in\mathbb{N}_0\times\mathbb{R}$ if $X$ is DRHR;

(c) $\overline{G}^{k*}(x)$ is $TP_2$ in $(k,x)\in\mathbb{N}_0\times\mathbb{R}$ if $X$ is IFR.

## 2 Two-parameter compound Poisson distributions

Ma[9] introduced the following two-parameter compound Poisson distribution:

$$Q(x;\theta,\nu,G)=\mathrm{e}^{-\theta}\sum_{k=0}^{\infty}\frac{\theta^k}{k!}G^{(\nu+rk)*}(x) \quad (2)$$

with two parameters $\theta\geqslant 0$ and $\nu$ a nonnegative integer, where $r$ is a given positive integer, and $G$ is a distribution function. Clearly, $Q(x;\theta,\nu,G)$ represents the distribution function of the random sum

$$S_{\nu+rN(\theta)}=\sum_{k=1}^{\nu}X_k+\sum_{k=1}^{rN(\theta)}X_{\nu+k} \quad (3)$$

where $\{X_i,\ i\geqslant 1\}$ is a sequence of i.i.d. random variables with distribution $G$, $N(\theta)$ is a Poisson random variable with parameter $\theta$ and independent of $\{X_i,\ i\geqslant 1\}$.

**Theorem 2.1** Let $G$ have a decreasing density function with $G(0)=0$. Then

(a) $Q(x;\theta,\nu,G)$ is log-concave in $x\in\mathbb{R}$ for all integer $\nu\in\mathbb{N}_0$ and $\theta\in\mathbb{R}_+$;

(b) $Q(x;\theta,\nu,G)$ is log-concave in $\nu\in\mathbb{N}_0$ for all $(x,\theta)\in\mathbb{R}\times\mathbb{R}_+$;

(c) $Q(x;\theta,\nu,G)$ is log-concave in $\theta\in\mathbb{R}_+$ for all $(x,\nu)\in\mathbb{R}\times\mathbb{N}_0$;

(d) $Q(x;\theta,\nu,G)$ is $TP_2$ both in $(x,\nu)\in\mathbb{R}\times\mathbb{N}_0$ and in $(x,\theta)\in\mathbb{R}\times\mathbb{R}_+$.

**Proof** (a) For each pair $(\nu,\theta)$,

$$Q(\cdot;\theta,\nu,G)=Q(\cdot;0,\nu,G)*Q(\cdot;\theta,0,G) \quad (4)$$

where $Q(\cdot;0,\nu,G)=G^{\nu*}(\cdot)$, and $*$ denotes the convolution operation. Since $G$ has a decreasing density, $G$ is DRHR by the definition, which, by Ref.[8, Corollary 1.B.63], implies that $Q(\cdot;0,\nu,G)$ is DRHR. On the other hand, for $\theta>0$, it follows from (3) that

$$S_{rN(\theta)}=\sum_{k=1}^{rN(\theta)}X_{\nu+k}=\sum_{j=1}^{N(\theta)}\Big(\sum_{k=1}^{r}X_{\nu+(j-1)r+k}\Big),$$

where $N(\theta)$ and $\{X_i,\ i\geqslant 1\}$ are the same as those in (3). It can be checked that if the density function $g$ of $G$ is decreasing, then

$$g^{r*}(x)=\int_{-\infty}^{\infty}g(x-u)g^{(r-1)*}(u)\,\mathrm{d}u$$

is also decreasing in $x$. Since $\sum_{k=1}^{r}X_{\nu+(j-1)r+k}$, $j\in\mathbb{N}$, are i.i.d. nonnegative and have decreasing density, by Lemma 1.3, we know that $Q(x;\theta,0,G)$ is DRHR for each $\theta>0$. It is trivial that $Q(x;0,0,G)$ is DRHR. So $Q(\cdot;\theta,0,G)$ is DRHR for $\theta\in\mathbb{R}_+$. Again, by Ref.[8, Corollary 1.B.63], it follows from (4) that $Q(\cdot;\theta,\nu,G)$ is also DRHR or, equivalently, $Q(\cdot;\theta,\nu,G)$ is log-concave in $x\in\mathbb{R}$. This proves part (a).

(b)~(d) Observe that $Q(\cdot;\theta,\nu,G)$ has RP($\theta$) and RP($\nu$). The desired results in parts (b)~(d) now follow from Lemma 1.1 directly. This completes the proof of the theorem.

**Theorem 2.2** Suppose that $G(0)=0$.

(a) If $G$ is ILR and $\nu\geqslant 1$, then the density function, $q(x;\theta,\nu,G)$, of $Q(x;\theta,\nu,G)$ is $TP_2$ in $(x,\theta)\in\mathbb{R}\times\mathbb{R}_+$, $RR_2$ in $(\theta,\nu)\in\mathbb{R}_+\times\mathbb{N}$, and is log-concave in $\theta\in\mathbb{R}_+$.

(b) If $G$ is DRHR, then $Q(x;\theta,\nu,G)$ is $TP_2$ in $(x,\theta)\in\mathbb{R}\times\mathbb{R}_+$, $RR_2$ in $(\theta,\nu)\in\mathbb{R}_+\times\mathbb{N}_0$, and is log-concave in $\theta\in\mathbb{R}_+$.

(c) If $G$ is IFR, then $\overline{Q}(x;\theta,\nu,G)$ is $TP_2$ in $(x,\theta)\in\mathbb{R}\times\mathbb{R}_+$, $RR_2$ in $(\theta,\nu)\in\mathbb{R}_+\times\mathbb{N}$, and is log-concave in $\theta\in\mathbb{R}_+$.

**Proof** We give the proof of part (a) only; the proofs of parts (b) and (c) are similar. First, note that when $\nu\geqslant 1$, the density function $q(x;\theta,\nu,G)$ of $Q(x;\theta,\nu,G)$ exists and is given by

$$q(x;\theta,\nu,G)=\mathrm{e}^{-\theta}\sum_{k=0}^{\infty}\frac{\theta^k}{k!}g^{(\nu+rk)*}(x) \quad (5)$$

where $g$ is the density function of $G$. Since $g^{(\nu+rk)*}(x)$ is $TP_2$ in $(k,x)\in\mathbb{N}_0\times\mathbb{R}$ by part (a) of Lemma 1.4, and $\theta^k/k!$ is $TP_2$ in $(\theta,k)\in\mathbb{R}_+\times\mathbb{N}_0$, applying the basic composition formula (cf. Ref. [3:17]) in Eq. (5) yields that $q(x;\theta,\nu,G)$ is $TP_2$ in $(x,\theta)\in\mathbb{R}\times\mathbb{R}_+$. On the other hand, since $q(x;\theta,\nu,G)$ has RP($\theta$) and RP($\nu$), it follows from Lemmas 1.1 and 1.2 that $q(x;\theta,\nu,G)$ is log-concave in $\theta\in\mathbb{R}_+$ for each $x$, and that $q(x;\theta,\nu,G)$ is $RR_2$ in $(\theta,\nu)\in\mathbb{R}_+\times\mathbb{N}$. This completes the proof of the theorem.

## 3 One application

Let $\chi^2_{n,\theta}$ denote the noncentral chi-square distribution with $n$ degrees of freedom and noncentral parameter $\theta$, where $(n,\theta)\in\mathbb{R}^2_+$. We adopt the convention that $\chi^2_n\equiv\chi^2_{n,0}$, the central chi-square distribution with $n$ degrees of freedom. The density function of $\chi^2_n$ is given by

$$h(x\mid n)=\frac{e^{-x/2}x^{n/2-1}}{2^{n/2}\Gamma(n/2)},\ x\in\mathbb{R}_+ \tag{6}$$

For $n=0$, $\chi^2_{0,\theta}$ is called the purely eccentric part by Hjort[10], while $\chi^2_{0,0}$ is the degenerate distribution with all mass at 0, and for $\theta>0$, $\chi^2_{0,\theta}$ puts mass $e^{-\theta/2}$ at 0. Ref. [5, Theorem 3.4] proved that $\chi^2_{n,\theta}$ is DRHR for all $(n,\theta)\in\mathbb{R}^2_+$ by using a more refined technique. They first proved that the density function of the continuous part of $\chi^2_{0,\theta}$ is log-concave, and then proved that $\chi^2_{0,\theta}$ is DRHR. We now present a different, but simple, proof by using Lemma 1.3.

### 3.1 Special case: $n\in\mathbb{N}_0$

First consider the case $n\in\mathbb{N}_0$, and let $X_{n,\theta}\sim\chi^2_{n,\theta}$. Then, $X_{n,\theta}$ has the following stochastic representation:

$$X_{n,\theta}=\sum_{i=1}^{n}X_i+\sum_{k=1}^{2N(\theta/2)}X_{n+k} \tag{7}$$

where $\{X_i,\ i\geqslant 1\}$ is a sequence of i. i. d. $\chi^2_1$-distributed random variable, and $N(\theta/2)$ is Poisson random variable with parameter $\theta/2$, independent of $\{X_i,\ i\geqslant 1\}$. Let $H(\cdot\mid n,\theta)$ denote the distribution function of $\chi^2_{n,\theta}$. From Eq. (6), it follows that the density function of $\chi^2_m$ is decreasing when $m\in[0,2]$. Applying Theorem 2.1 to Eq. (6), we conclude that

① $H(x\mid n,\theta)$ is log-concave in $x\in\mathbb{R}_+$;

② $H(x\mid n,\theta)$ is $TP_2$ in $(x,n)\in\mathbb{R}_+\times\mathbb{N}_0$ and $TP_2$ in $(x,\theta)\in\mathbb{R}^2_+$;

③ $H(x\mid n,\theta)$ is log-concave in $n\in\mathbb{N}_0$ for all $(x,\theta)\in\mathbb{R}^2_+$;

④ $H(x\mid n,\theta)$ is log-concave in $\theta\in\mathbb{R}_+$ for all $(x,n)\in\mathbb{R}_+\times\mathbb{N}_0$.

### 3.2 General case: $n\in\mathbb{R}_+$

**Theorem 3.1**[5] $\chi^2_{n,\theta}$ is DRHR for all $(n,\theta)\in\mathbb{R}^2_+$.

**Proof** First, consider the case $\theta=0$. Note that $\chi^2_n$ is ILR for $n\geqslant 2$, and that $\chi^2_n$ has a decreasing density for $n\in(0,2)$ and $\chi^2_0$ is the degenerate distribution with all mass at zero. Then $\chi^2_n$ is DRHR for all $n\in\mathbb{R}_+$.

Next, consider the case $\theta>0$. Let $X_{n,\theta}$ be a random variable with distribution $\chi^2_{n,\theta}$, and $N(\theta/2)$ be a Poisson random variable with parameter $\theta/2$. For $n=0$, it follows from Ref. [10] that

$$X_{0,\theta}\overset{d}{=}\sum_{k=1}^{N(\theta/2)}Y_k,$$

where $\overset{d}{=}$ means equality in distribution, and $\{Y_k,\ k\geqslant 1\}$ is a sequence of i. i. d. $\chi^2_2$-distributed random variables, independent of $N(\theta/2)$. Since the density of $\chi^2_2$ is decreasing, by Lemma 1.3, we get that $\chi^2_{0,\theta}$ is DRHR. For $n\in(0,\infty)$, from Ref. [11], we have

$$\chi^2_{n,\theta}=\chi^2_n*\chi^2_{0,\theta} \tag{8}$$

Note that $\chi^2_n$ is DRHR for all $n>0$. Applying Ref. [8, Corollary 1.B.63] in Eq. (8) yields that $\chi^2_{n,\theta}$ is DRHR. This completes the proof.

From Ref. [11], it also follows that

$$\chi^2_{n_1+n_2,\theta}=\chi^2_{n_1,\theta}*\chi^2_{n_2}\ \text{for all}\ (n_1,n_2,\theta)\in\mathbb{R}^3_+ \tag{9}$$

and

$$\chi^2_{n,\theta_1+\theta_2}=\chi^2_{n,\theta_1}*\chi^2_{0,\theta_2}\ \text{for all}\ (n,\theta_1,\theta_2)\in\mathbb{R}^3_+ \tag{10}$$

Let $H(\cdot\mid n,\theta)$ denote the distribution function of $\chi^2_{n,\theta}$. Then (9) and (10) state that $H(\cdot\mid n,\theta)$ has both RP($n$) and RP($\theta$). Based on these observations, we conclude from Lemmas 1.1 and 1.2 and Theorem 3.1 that (see Ref.[5, Theorem 3.9])

① $H(x \mid n,\theta)$ is $TP_2$ in $(x,n) \in \mathbb{R}_+^2$ and $TP_2$ in $(x,\theta) \in \mathbb{R}_+^2$;

② $H(x \mid n,\theta)$ is log-concave in $n \in \mathbb{R}_+$ for all $(x,\theta) \in \mathbb{R}_+^2$;

③ $H(x \mid n,\theta)$ is log-concave in $\theta \in \mathbb{R}_+$ for all $(x,n) \in \mathbb{R}_+^2$.

The main results in this short note can be used to establish the log-concave properties of F and beta distributions; interested readers could view Ref.[5].

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