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Proof of chromaticity of the complete tripartite graphs $K(n-k, n-3, n)$

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Abstract：Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. A graph G is chromatically unique if for any graph H，$P(H, \lambda)=P(G, \lambda)$ implies $G \cong H$. Here，by comparing the number of the triangular subgraphs and the number of the quadrangular subgraphs without chords，the chromatic uniqueness problem of the complete tripartite graphs $K(n-k, n-3, n)$ was completely solved. It was proved that $K(n-k, n-3, n)$ is chromatically unique if $n \geqslant k+2 \geqslant 5$.

Key words：complete tripartite graph；chromatically uniqueness；triangular subgraph；quadrangular subgraph without chords

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完全3部图$K(n-k, n-3, n)$色唯一性的证明

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摘要：设$P(G,\lambda)$是图G的色多项式，如果任意与图G的色多项式相等($P(G,\lambda)=P(H,\lambda)$)的图$H$都与图$G$同构($G \cong H$)，则称图$G$是色唯一图. 这里，通过比较图的三角形子图和无弦四边形的个数，完全解决了一类完全三部图$K(n-k, n-3, n)$的色唯一性问题，证明了，若$n \geqslant k+2 \geqslant 5$，则完全三部图$K(n-k, n-3, n)$是色唯一图.

关键词：完全三部图；色唯一性；三角形子图；无弦四边形子图

0 Introduction

All graphs considered here are finite and simple. For a graph G，let $V(G)$，$E(G)$ be the vertex set，edge set of G，respectively. We denote by $P(G,\lambda)$ the chromatic polynomial of G. Two graphs G and H are said to be chromatically equivalent，denoted by $G \sim H$ if $P(G,\lambda)=P(H,\lambda)$. A graph G is said to be chromatically unique or χ-unique if $G \cong H$ for any graph H such that $H \sim$

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G. The complete tripartite graph can be denoted by $K(n+a_1, n+a_2, n+a_3)$ having partite sets V_i with $|V_i|=n+a_i$ for $i=1,2,3$.

Ref. [1] gave the conjecture that the $K(n-k, n, n)$ is chromatically unique if $n\geqslant k+2\geqslant 4$. Refs. [2-4] showed that the $K(n-k, n-i, n)$ is chromatically unique if $n\geqslant k+2\geqslant 4$ and $i=0,1$. Ref. [5] showed that the $K(n-k, n-v, n)$ is chromatically unique if $k\geqslant v+2\geqslant 4$ and $n\geqslant v^2(k+v/3)/4$, and that the $K(n-k, n-2, n)$ is chromatically unique if $n\geqslant k+2\geqslant 4$. Ref. [6] showed that the $K(n-k, n-v, n)$ is chromatically unique if $4\geqslant v\geqslant 2, k\geqslant v$ and $n\geqslant k^2/4+v+1$. Ref. [7] showed that $K(n-k, n-v, n)$ is chromatically unique, if $k-v\geqslant 2, v\geqslant 1$ and $n-v\geqslant \max\{[(k-v)^2/4+k], [(k-v)^2/4+3(k-v)/2+2v-11/4], [(k-v)v+k-2v+1]\}$.

Here we show that the complete tripartite graph $K(n-k, n-3, n)$ is chromatically unique for $n\geqslant k+2\geqslant 5$.

For a positive integer i, a partition $\{U_1, U_2, \cdots, U_i\}$ of $V(G)$ is called a i-independent partition in G if each U_i is a nonempty independent set of G. By $m(G, i)$ we denote the number of i-independent partition in G.

Let A_1, A_2, A_3 be the edge subsets of $K(n+a_1, n+a_2, n+a_3)$ with $A_1\subseteq\{xy \mid \forall x\in V_2, \forall y\in V_3\}$, $A_2\subseteq\{xy \mid \forall x\in V_1, \forall y\in V_3\}$ and $A_3\subseteq\{xy \mid \forall x\in V_1, \forall y\in V_2\}$. Let $A=A_1\cup A_2\cup A_3$, $|A_1|=l_1$, $|A_2|=l_2$, $|A_3|=l_3$. By $K(n+a_1, n+a_2, n+a_3)-A$ we denote the graph obtained by deleting all edges in A from the complete tripartite graph $K(n+a_1, n+a_2, n+a_3)$. By $\Delta(G)$ we denote the number of triangles C_3 in G. By $\alpha(G)$ we denote the number of cycles C_4 of without chords in G. By $T(e, r)$ we denote the tree T_3 containing the edges e and r in G. By $\Delta(e, r, w)$ we denote the triangle C_3 containing the edges e, r and w in G. Let

$$\varphi=|\{T(e,r) \mid e\in A_1, r\in A_2\}|+|\{T(e,r) \mid e\in A_1, r\in A_3\}|+|\{T(e,r) \mid e\in A_2, r\in A_3\}|,$$

$$\varphi_1=|\{\Delta(e,r,w) \mid e\notin A_1, r\in A_2, w\in A_3\}|,$$

$$\varphi_2=|\{\Delta(e,r,w) \mid e\in A_1, r\notin A_2, w\in A_3\}|,$$

$$\varphi_3=|\{\Delta(e,r,w) \mid e\in A_1, r\in A_2, w\notin A_3\}|,$$

$$\gamma=|\{\Delta(e,r,w) \mid e\in A_1, r\in A_2, w\in A_3\}|.$$

Clearly $\varphi=\varphi_1+\varphi_2+\varphi_3+3\gamma$.

1 Some lemmas

Lemma 1.1[8] Let $K(n+a_1, n+a_2, n+a_3)-A\sim K(n-k, n-v, n)$, $n>k\geqslant v\geqslant 0$. Then

① $a_1+a_2+a_3=-(k+v)$,

② $|A|=a_1a_2+a_1a_3+a_2a_3-vk=(k^2+v^2-a_1^2-a_2^2-a_3^2)/2$,

③ $\Delta(K(n+a_1, n+a_2, n+a_3)-A)=\Delta(K(n-k, n-v, n))$,

④ $\alpha(K(n+a_1, n+a_2, n+a_3)-A)=\alpha(K(n+a_1, n+a_2, n+a_3))$.

Lemma 1.2[2] Let $K(n+a_1, n+a_2, n+a_3)-A\sim K(n-k, n-v, n)$, $0\leqslant v\leqslant k$, $a_1\leqslant a_2\leqslant a_3$. Then $-v\leqslant a_3\leqslant 0$.

Lemma 1.3[9] Let $a_1+a_2+a_3=-(k+v)$, $|A|=a_1a_2+a_1a_3+a_2a_3-vk$ and

$$\bar{\Delta}=\Delta(K(n+a_1, n+a_2, n+a_3)-A)-\Delta(K(n-k, n-v, n)).$$

Then

$$\bar{\Delta}=a_1a_2a_3-a_1l_1-a_2l_2-a_3l_3+\varphi-\gamma.$$

Lemma 1.4[5] Let $K(n+a_1, n+a_2, n+a_3)-A\sim K(n-k, n-v, n)$, $a_1\leqslant a_2\leqslant a_3$, $n\geqslant k+2\geqslant v+2$, $\bar{\alpha}=\alpha(K(n+a_1, n+a_2, n+a_3)-A)-\alpha(K(n-k, n-v, n))$. Then

① $\bar{\alpha}\leqslant-(l_1l_2+l_1l_3+l_2l_3)/2+(\varphi-\gamma)/4-(n+a_2+a_3-a_1)\varphi_1/2-(n+a_1+a_3-a_2)\varphi_2/2-(n+a_1+a_2-a_3)\varphi_3/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(\varphi_1+\varphi_2+\varphi_3)\gamma-(3\gamma+1)\gamma/2$,

② $\bar{\alpha}\leqslant-(\varphi-\gamma)/4-(n+a_2+a_3-a_1)\varphi_1/2-(n+a_1+a_3-a_2)\varphi_2/2-(n+a_1+a_2-a_3)\varphi_3/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(\varphi_1+\varphi_2+\varphi_3)\gamma-(3\gamma+1)\cdot\gamma/2-(n+a_3-1)(n+a_2-(l_1+1))l_1/2-(n+a_3-1)(n+a_1-(l_2+1))l_2/2$,

③ $\bar{\alpha}\leqslant-(\varphi-\gamma)/4-(n+a_2+a_3-a_1)\varphi_1/2-(n+a_1+a_3-a_2)\varphi_2/2-(n+a_1+a_2-a_3)\varphi_3/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(\varphi_1+\varphi_2+\varphi_3)\gamma-(3\gamma+1)\cdot\gamma/2-(n+a_3-1)(n+a_2-(l_1+1))l_1/2$,

④ $\bar{\alpha}\leqslant-(\varphi-\gamma)/4-(n+a_2+a_3-a_1)\varphi_1/2-(n+a_1+a_3-a_2)\varphi_2/2-(n+a_1+a_2-a_3)\varphi_3/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(\varphi_1+\varphi_2+\varphi_3)\gamma-(3\gamma+1)\cdot$

$\gamma/2-(n+a_3-1)(n+a_1-(l_2+1))l_2/2$.

Lemma 1.5[10] Let $K(n+a_1,n+a_2,n+a_3)-A_3\sim K(n+b_1,n+b_2,n+b_3)$, $a_3\in\{b_1,b_2,b_3\}$ and $\min\{n+b_1,n+b_2,n+b_3\}\geqslant 2$. Then

$$K(n+a_1,n+a_2,n+a_3)-A_3\cong K(n+b_1,n+b_2,n+b_3)\ ,\ |A_3|=0.$$

Lemma 1.6[8] Let $G\sim H$. Then $m(G,i)=m(H,i)$ for $i=1,2,\cdots,|V(G)|$.

2 Main result

Theorem 2.1 The complete tripartite graph $K(n-k,n-3,n)$ is chromatically unique if $n\geqslant k+2\geqslant 5$.

Proof Let $K(n+a_1,n+a_2,n+a_3)-A\sim K(n-k,n-3,n)$ and $a_1\leqslant a_2\leqslant a_3$. By Lemmas 1.1 and 1.2, we have

$$a_1+a_2+a_3=-(k+3),$$
$$|A|=a_1a_2+a_1a_3+a_2a_3-3k,$$
$$\bar{\Delta}=0,\ \bar{\alpha}=0,\ -3\leqslant a_3\leqslant 0.$$

By Lemma 1.3,

$$\bar{\Delta}=a_1a_2a_3-a_1l_1-a_2l_2-a_3l_3+\varphi-\gamma=a_3(k+a_3)(3+a_3)-(a_1-a_3)l_1-(a_2-a_3)l_2+\varphi-\gamma=0.$$

Since $l_1l_2\geqslant\varphi_3+\gamma$, $l_1l_3\geqslant\varphi_2+\gamma$ and $l_2l_3\geqslant\varphi_1+\gamma$, we have that $l_1l_2+l_1l_3+l_2l_3\geqslant\varphi$.

Now we consider the following three cases.

Case 1 $-3<a_3<0$.

Suppose $l_1+l_3=0$, we have $\varphi-\gamma=0$. Hence $a_1a_2a_3-a_2l_2=0$, $l_2=a_1a_3=|A|=a_1a_3-(k+a_2)(3+a_2)$ and $(k+a_2)(3+a_2)=0$.

Hence $a_2=-3$ or $a_2=-k$. From Lemma 1.5, we have $|A|=0$, $K(n+a_1,n+a_2,n+a_3)-A\cong K(n-k,n-3,n)$. Hence $\{a_1,a_2,a_3\}=\{-k,-3,0\}$, a contradicting $a_3<0$. Such $l_1+l_3\neq 0$.

Similarly, we have that $l_1+l_2\neq 0$ and $l_2+l_3\neq 0$. Hence $l_1l_2+l_1l_3+l_2l_3\neq 0$. Since $a_3=-1$ or $a_3=-2$, we have

$$a_3(k+a_3)(3+a_3)=-2(k+a_3),$$
$$\bar{\Delta}=-2(k+a_3)-(a_1-a_3)l_1-(a_2-a_3)l_2+\varphi-\gamma=0.$$

Suppose $\varphi-\gamma=0$,From ① of Lemma 1.4, we have $\bar{\alpha}<0$, contradicting $\bar{\alpha}=0$.

For $\varphi-\gamma>0$, we consider the following three subcases.

Subcase 1.1 $l_1\geqslant 2$. We consider the following three subcases.

Subcase 1.1.1 $l_2\geqslant 2$. Since $a_1+a_2-2a_3=a_1+a_2+a_3-3a_3=-(k+3)-3a_3$, we have

$$\bar{\Delta}=-2(k+a_3)-2(a_1+a_2-2a_3)-(a_1-a_3)(l_1-2)-(a_2-a_3)(l_2-2)+\varphi-\gamma=0,$$
$$6+4a_3=(a_1-a_3)(l_1-2)+(a_2-a_3)(l_2-2)-(\varphi-\gamma)\leqslant 0.$$

Hence $a_3=-2$ and $\varphi-\gamma\leqslant 2$. Since $\varphi-\gamma\geqslant 2\gamma$, we get $\gamma\leqslant 1$. Hence

$$n+a_1+a_2-a_3-(\varphi-\gamma+1)\geqslant(k+2-(k+3)-2a_3)-3=0.$$

Since $|A|=(k^2+3^2-a_1{}^2-a_2{}^2-a_3{}^2)/2\geqslant 4$, we have $a_1<a_3=-2$.

Suppose $l_1\geqslant 3$. Since $\varphi-\gamma=\varphi_1+\varphi_2+\varphi_3+2\gamma\leqslant-6-4a_3+(a_1-a_3)\leqslant 1$, we have $\gamma=0$. By ① of Lemma 1.4,we have

$$\bar{\alpha}<-(n+a_1+a_2-a_3)(\varphi_1+\varphi_2+\varphi_3)/2+(\varphi-\gamma+1)(\varphi-\gamma)/2\leqslant-(\varphi+1)\varphi/2+(\varphi+1)\varphi/2=0,$$

contradicting $\bar{\alpha}=0$.

Suppose $l_1=2$. Since $n+a_3-1\geqslant k+2-3\geqslant 2\geqslant 2\gamma$,

$$n+a_2-(l_1+1)=n+(a_1+a_2+a_3)-(a_1+a_3)-3\geqslant n-(k+3)+5-3\geqslant 1,$$

by ③ of Lemma 1.4, we have

$$\bar{\alpha}<-(n+a_1+a_2-a_3)(\varphi_1+\varphi_2+\varphi_3)/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(\varphi_1+\varphi_2+\varphi_3)\gamma-(3\gamma+1)\gamma/2-(n+a_3-1)(n+a_2-(l_1+1))l_1/2\leqslant-(\varphi-\gamma+1)(\varphi_1+\varphi_2+\varphi_3)/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(\varphi_1+\varphi_2+\varphi_3)\gamma-(3\gamma+1)\gamma/2-\gamma\leqslant(\varphi-\gamma+1)\gamma-(\varphi-3\gamma)\gamma-(3\gamma+1)\gamma/2-\gamma=2\gamma^2-(3\gamma+1)\gamma/2\leqslant 0,$$

contradicting $\bar{\alpha}=0$.

Subcase 1.1.2 $l_2=1$. Since

$$\bar{\Delta}=-(k-3-a_3)-(a_1-a_3)(l_1-1)-(a_2-a_3)(l_2-1)+\varphi-\gamma=0,$$

we have $\varphi-\gamma\leqslant(a_1-a_3)+(k-3-a_3)$,

$$(n+a_1-2)-(\varphi-\gamma)\geqslant n+a_1-2-(k-3-a_3)-(a_1-a_3)\geqslant 3+2a_3,$$
$$(n+a_3-1)-(\varphi-\gamma+1)=n+a_3-2-(\varphi-\gamma)\geqslant 3+2a_3.$$

Suppose $a_3=-1$. We have $(n+a_1-2)-(\varphi-\gamma)>0$, $(n+a_3-1)-(\varphi-\gamma+1)>0$. From ④ of Lemma 1.4, we have

$$\bar{\alpha}<-(n+a_3-1)(n+a_1-(l_2+1))l_2/2+(\varphi-\gamma+1)(\varphi-\gamma)/2\leqslant -(\varphi-\gamma+1)(n+a_1-2)/2+(\varphi-\gamma+1)(\varphi-\gamma)/2<0,$$

contradicting $\bar{\alpha}=0$.

Suppose $a_3=-2$. We have

$$n+a_1+a_2-a_3=n+a_1+a_2+a_3-2a_3=n-(k+3)+4=(n-k)+1\geqslant 3,$$
$$n+a_1-2-(\varphi-\gamma)\geqslant -1,$$
$$(n+a_3-1)-(\varphi-\gamma+1)\geqslant -1.$$

From ④ of Lemma 1.4, we get

$$\bar{\alpha}<-(n+a_2+a_3-a_1)\varphi_1/2-(n+a_1+a_3-a_2)\varphi_2/2-(n+a_1+a_2-a_3)\varphi_3/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(3\gamma+1)\gamma/2-(n+a_3-1)(n+a_1-2)/2\leqslant -3(\varphi_1+\varphi_2+\varphi_3)/2-(3\gamma+1)\gamma/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(\varphi-\gamma-1)(\varphi-\gamma)/2\leqslant -3(\varphi_1+\varphi_2+\varphi_3)/2-2\gamma+(\varphi-\gamma)\leqslant 0,$$

contradicting $\bar{\alpha}=0$.

Subcase 1.1.3 $l_2=0$. Such $\varphi=\varphi_2$ and $\gamma=0$. Since $\bar{\Delta}=-2(k+a_3)-(a_1-a_3)l_1+(\varphi-\gamma)=0$, we have

$$(n+a_1+a_3-a_2)-(\varphi_2+1)\geqslant (n+a_1+a_3-a_2)-2(k+a_3)-2(a_1-a_3)-1=(n+a_3-a_1-a_2)-2k-1\geqslant (k+2)+2a_3+(k+3)-2k-1=4+2a_3\geqslant 0.$$

From ① of Lemma 1.4, we have $\bar{\alpha}<0$, contradicting $\bar{\alpha}=0$.

Subcase 1.2 $l_1=1$. we consider the following three subcases.

Subcase 1.2.1 $l_2\geqslant 2$. Since

$$\bar{\Delta}=-(k-3-a_3)-(a_1-a_3)(l_1-1)-(a_2-a_3)(l_2-1)+\varphi-\gamma=0,$$

we have

$$(n+a_2-2)-(\varphi-\gamma)\geqslant n+a_2-2-(k-3-a_3)-(a_2-a_3)\geqslant k+2+2a_3+1-k=3+2a_3,$$
$$(n+a_3-1)-(\varphi-\gamma+1)=(n+a_3-2)-(\varphi-\gamma)\geqslant 3+2a_3.$$

Suppose $a_3=-1$. We have

$$(n+a_2-2)-(\varphi-\gamma)>0,$$
$$(n+a_3-1)-(\varphi-\gamma+1)>0.$$

By ③ of Lemma 1.4, we have

$$\bar{\alpha}<(\varphi-\gamma+1)(\varphi-\gamma)/2-(n+a_3-1)(n+a_2-2)/2\leqslant 0,$$

contradicting $\bar{\alpha}=0$.

Suppose $a_3=-2$. We have

$$(n+a_2-2)-(\varphi-\gamma)>-1,$$
$$(n+a_3-1)-(\varphi-\gamma+1)>-1,$$
$$n+a_1+a_2-a_3\geqslant 3.$$

From ③ of Lemma 1.4, we have

$$\bar{\alpha}<-(n+a_2+a_3-a_1)\varphi_1-(n+a_1+a_3-a_2)\varphi_2-(n+a_1+a_2-a_3)\varphi_3+(\varphi-\gamma+1)(\varphi-\gamma)/2-(3\gamma+1)\gamma/2-(n+a_3-1)(n+a_2-(l_1+1))l_1/2\leqslant -3(\varphi_1+\varphi_2+\varphi_3)/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(3\gamma+1)\gamma/2-(\varphi-\gamma)(\varphi-\gamma-1)/2\leqslant -3(\varphi_1+\varphi_2+\varphi_3)/2+(\varphi-\gamma)-2\gamma\leqslant 0,$$

contradicting $\bar{\alpha}=0$.

Subcase 1.2.2 $l_2=1$. Since $\bar{\Delta}=-(k-3-a_3)+\varphi-\gamma=0$, we have

$$[n+a_2-(l_1+1)]l_1+[n+a_1-(l_2+1)]l_2-(\varphi-\gamma)=2n+a_1+a_2-4-(\varphi-\gamma)\geqslant 2(k+2)-a_3-(k+3)-(k-3-a_3)-4=0,$$
$$(n+a_3-1)-(\varphi-\gamma)=(n+a_3-1)-(k-3-a_3)=n-k+2+2a_3\geqslant 4+2a_3\geqslant 0.$$

Since $n+a_1+a_2-a_3=n-(k+3)-2a_3\geqslant 1$, by ② of Lemma 1.4, we have

$$\bar{\alpha}<-(n+a_1+a_2-a_3)(\varphi_1+\varphi_2+\varphi_3)/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(3\gamma+1)\gamma/2-(n+a_3-1)[n+a_2-(l_1+1)]l_1/2-(n+a_3-1)[n+a_1-(l_2+1)]l_2/2\leqslant -(\varphi_1+\varphi_2+\varphi_3)/2+(\varphi-\gamma+1)(\varphi-\gamma)/2-(3\gamma+1)\gamma/2-(\varphi-\gamma)(\varphi-\gamma)/2\leqslant -(\varphi-3\gamma)/2+(\varphi-\gamma)/2-(3\gamma+1)\gamma/2=\gamma-(3\gamma+1)\gamma/2=-(3\gamma-1)\gamma/2\leqslant 0,$$

contradicting $\bar{\alpha}=0$.

Subcase 1.2.3 $l_2=0$. Such $\gamma=0$, $\varphi=\varphi_2$. Since $m(K(n+a_1,n+a_2,n+a_3)-A,3)=m(K(n-k,n-3,n),3)=1$, we have that the bipartite subgraph $K(n+a_1,n+a_2)-A_3$ is connected. Hence $\varphi_2\leqslant n+a_1-1$,

$$(n+a_1+a_3-a_2)-(\varphi_2+1)\geqslant (n+a_1+a_3-a_2)- (n+a_1-1)-1=a_3-a_2\geqslant 0.$$

By ① of Lemma 1.4, we have

$$\bar{\alpha}<-(n+a_1+a_3-a_2)\varphi_2/2+ (\varphi_2+1)\varphi_2/2\leqslant 0,$$

contradicting $\bar{\alpha}=0$.

Subcase 1.3 $l_1=0$. Such $\varphi-\gamma=\varphi_1$, $\gamma=0$. We consider the following two subcases.

Subcase 1.3.1 $l_2\geqslant 2$. Since $\bar{\Delta}=-2(k+a_3)-(a_2-a_3)l_2+\varphi-\gamma=0$, we have

$$(n+a_2+a_3-a_1)-(\varphi_1+1)\geqslant (n+a_2+a_3-a_1)- 2(k+a_3)-2(a_2-a_3)-1= n+a_3-a_1-a_2-2k-1= n+(k+3)+2a_3-2k-1\geqslant (k+2)+(k+3)+2a_3-2k-1=4+2a_3\geqslant 0.$$

By ① of Lemma 1.4, we have

$$\bar{\alpha}<-(n+a_2+a_3+a_1)\varphi_1/2-(\varphi_1+1)\varphi_1/2\leqslant 0,$$

contradicting $\bar{\alpha}=0$.

Subcase 1.3.2 $l_2=1$. Since that bipartite subgraph $K(n+a_1,n+a_2)-A_3$ is connected, we have $\varphi_1\leqslant n+a_2-1$. Hence

$$(n+a_2+a_3-a_1)-(\varphi_1+1)\geqslant (n+a_2+a_3-a_1)- (n+a_2-1)-1=a_3-a_1\geqslant 0.$$

By ① of Lemma 1.4, we have $\bar{\alpha}<0$, contradicting $\bar{\alpha}=0$.

Case 2 $a_3=-3$.

Similarly $l_1l_2+l_1l_3+l_2l_3\neq 0$. Since $\bar{\Delta}=-(a_1-a_3)l_1-(a_2-a_3)l_2+\varphi-\gamma=0$, we have $\varphi-\gamma=0$. By ① of Lemma 1.4, we have $\bar{\alpha}<0$, contradicting $\bar{\alpha}=0$.

Case 3 $a_3=0$.

By Lemma 1.1, we have $|A|=a_1a_2-3k\geqslant 0$. Hence $a_1a_2\neq 0$. Since $-a_1l_1\geqslant 0$, $-a_2l_2\geqslant 0$, $-a_1l_1-a_2l_2+\varphi-\gamma=0$, we have $l_1=l_2=0$. By Lemma 1.5, we have

$$K(n+a_1,n+a_2,n+a_3)-A\cong K(n-k,n-3,n).$$

Hence the complete tripartite graph $K(n-k,n-3,n)$ is chromatically unique for $n\geqslant k+2\geqslant 5$.

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