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# Stability of the backward problem of parabolic equation with time-dependent reaction coefficient

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**Abstract**: The backward problem of two-dimensional parabolic equation with a time-dependent coefficient was considered. This problem is severely ill-posed, i. e. , the solution(if it exists) does not depend continuously on the given data. Using the method of regularization, an optimal stability estimation of the solution was derived. A numerical example shows the effectiveness of the presented method.

**Key words**: backward problem; ill-posed problem; regularization; error estimate

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# 带有时间独立反应系数的抛物方程逆时问题的稳定性

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**摘要**:考虑了带有时间独立系数的二维抛物方程的逆时问题.此问题具有严重的不适定性,即解(若存在)不连续依赖于给定的数据.利用正则化方法,得到了解的最优稳定估计.数值算例展示了该正则化方法的有效性.

**关键词**:逆时问题;不适定问题;正则化;误差估计

## 0 Introduction

We discuss the backward problem of nonhomogeneous two-dimensional parabolic equation with a time-dependent coefficient:

$$\left.\begin{array}{l} u_t(x,y,t)-a(t)\Delta u=f(x,y,t), \\ (x,y,t)\in B_{r_0}\times(0,T] \end{array}\right\} \quad (1)$$

$$u|_{\partial B_{r_0}}=0,\ t\in[0,T] \quad (2)$$

$$u(x,y,T)=g(x,y),(x,y)\in B_{r_0} \quad (3)$$

where $a(t)$ , $f(x,y,t)$ and $g(x,y)$ are given continuous functions. $B_{r_0}$ is the circle domain of radius $r_0$ centered at the origin. We assume that there are constants $p,q$ such that

$$0<p\leqslant a(t)\leqslant q \quad (4)$$

When we know the information $u(x,y,t)$ at $t=T$ , we need to obtain the information of $u(x,$

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$y, t)$ at $t=0$. This problem is called backward heat conduction problem (BHCP). Since the solution (if it exists) to BHCP does not depend continuously on the given data[1], a small perturbation on the data can greatly affect the exact solution. Hence, it is impossible to solve the BHCP using classical methods. The method of quasi-reversibility (QR) proposed in Ref. [2] was designed to solve ill-posed problems. However, since the order of the operator is replaced by an operator of second order with a small parameter, it is very difficult to perform the numerical implementation. The method of logarithmic convexity[3] is also effective to get the conditional stability estimate when $0 < t \leqslant T$ rather than $t=0$. Furthermore, it usually requires $a'(t) \leqslant 0$. Some researchers have studied this problem using the boundary element method[4-6]. A group preserving scheme was introduced in Ref. [7]. Some regularization methods[8-12] were introduced to solve the BHCP. These methods were demonstrated to be very effective, whereas most of them were applied to the linear homogeneous case or constant coefficient and one-dimensional problems. Ref. [12] gave a modified method for regularizing the backward problem of one-dimensional parabolic equation with the time-dependent coefficient and obtained the error estimates between the regularized solution and the exact solution as follows

$$\| u(\cdot, t) - u^{\epsilon}(g_{\epsilon})(\cdot, t) \| \leqslant \epsilon^{\frac{t+\gamma}{T+\gamma}} + A_2 \epsilon^{\frac{p\gamma}{q^2(T+\gamma)}} \tag{5}$$

where $\gamma \in (0, qT)$. We can easily see that the above estimate tends to zero slowly in a neighborhood of zero when $\frac{p}{q^2}$ is very small.

In this paper, we consider the two-dimensional equation with the time-dependent coefficient (1)~(3). To our best knowledge, few papers have been published related to the time-dependent coefficient and two-dimensional equation, and even fewer in the area is circular domain. We generalize the regularization method and obtain a better convergence stability than Ref. [12] with similar assumptions.

## 1 Expression of solution

In order to get the expression of the solution, we let $x = r\cos\theta$, $y = r\sin\theta$, then $B_{r_0}$ is transformed into $[0, r_0] \times [0, 2\pi)$. For convenience, the functions $u, f, g$ which have been transformed are still represented by $u, f, g$. Then Eqs. (1)~(3) become

$$u_t - a(t)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right] = f(r, \theta, t),\ 0 < r < r_0 \tag{6}$$

$$u|_{r=r_0} = 0 \tag{7}$$

$$u|_{t=T} = g(r, \theta) \tag{8}$$

Assuming $u = T(t)R(r)\Theta(\theta)$, and using the method of separation of variables, we obtain

$$\begin{cases} \Theta'' + \mu\Theta = 0, \\ \Theta(\theta + 2\pi) = \Theta(\theta), \end{cases}$$

$$\begin{cases} \frac{1}{r}(rR')' + (\lambda - \frac{\mu}{r^2})R = 0,\ 0 < r < r_0, \\ |R(0)| < +\infty,\ R(r_0) = 0, \end{cases}$$

and ordinary differential equation

$$T' + \lambda a(t) T = 0.$$

By solving the above eigenvalue problem, we can get the solution of the homogeneous equation:

$$v(r, \theta, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big[\exp(\omega_{mn}^2(A(T) - A(t))) \cdot (A_{mn}\cos m\theta + B_{mn}\sin m\theta) J_m(\omega_{mn} r)\Big] \tag{9}$$

where $\omega_{mn}$ is the $n$th positive root of Bessel function $J_m(\omega r_0)$ and $A(t) = \int_0^t a(s)\mathrm{d}s$. Using the impulse principle, we can get the solution of Eqs. (6)~(8).

$$u(r, \theta, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big[\exp(\omega_{mn}^2(A(T) - A(t)))(A_{mn}\cos m\theta + B_{mn}\sin m\theta) J_m(\omega_{mn} r) - \int_t^T \exp(\omega_{mn}^2(A(s) - A(t)))(C_{mn}(s)\cos m\theta + D_{mn}(s)\sin m\theta) J_m(\omega_{mn} r)\mathrm{d}s\Big] \tag{10}$$

where

$$A_{mn}=\frac{\delta_m}{\pi r_0^2 J_{m+1}^2(\omega_{mn}r_0)}\int_0^{r_0}\int_0^{2\pi} g(r,\theta)\cos m\theta\, J_m(\omega_{mn}r)r\,\mathrm{d}r\,\mathrm{d}\theta,$$

$$B_{mn}=\frac{2}{\pi r_0^2 J_{m+1}^2(\omega_{mn}r_0)}\int_0^{r_0}\int_0^{2\pi} g(r,\theta)\sin m\theta\, J_m(\omega_{mn}r)r\,\mathrm{d}r\,\mathrm{d}\theta,$$

$$C_{mn}(s)=\frac{\delta_m}{\pi r_0^2 J_{m+1}^2(\omega_{mn}r_0)}\int_0^{r_0}\int_0^{2\pi} f(r,\theta,s)\cos m\theta\, J_m(\omega_{mn}r)r\,\mathrm{d}r\,\mathrm{d}\theta,$$

$$D_{mn}(s)=\frac{2}{\pi r_0^2 J_{m+1}^2(\omega_{mn}r_0)}\int_0^{r_0}\int_0^{2\pi} f(r,\theta,s)\sin m\theta\, J_m(\omega_{mn}r)r\,\mathrm{d}r\,\mathrm{d}\theta,$$

and

$$\delta_m=\begin{cases}1, & m=0;\\ 2, & m>0.\end{cases}$$

Let

$$\bar{v}_{mn}(r,t)=\Big(\exp(\omega_{mn}^2(A(T)-A(t)))A_{mn}-\int_t^T \exp(\omega_{mn}^2(A(s)-A(t)))C_{mn}(s)\mathrm{d}s\Big)J_m(\omega_{mn}r) \tag{11}$$

$$\tilde{v}_{mn}(r,t)=\Big(\exp(\omega_{mn}^2(A(T)-A(t)))B_{mn}-\int_t^T \exp(\omega_{mn}^2(A(s)-A(t)))D_{mn}(s)\mathrm{d}s\Big)J_m(\omega_{mn}r) \tag{12}$$

and

$$\left.\begin{aligned}\phi_m(r,t)&=\sum_{n=1}^{\infty}\bar{v}_{mn}(r,t),\\ \psi_m(r,t)&=\sum_{n=1}^{\infty}\tilde{v}_{mn}(r,t)\end{aligned}\right\} \tag{13}$$

then

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\phi_m(r,t)\cos m\theta+\psi_m(r,t)\sin m\theta \tag{14}$$

Since the solution $u$ is not stable about the observed data $g$, we approximate Eqs. (6)～(8) by using the regularization problem

$$u^{\epsilon}(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big[\frac{\exp(-\omega_{mn}^2A(t))}{\beta+\exp(-\omega_{mn}^2A(T))}(A_{mn}\cos m\theta+B_{mn}\sin m\theta)J_m(\omega_{mn}r)-\int_t^T\frac{\exp(\omega_{mn}^2(A(s)-A(t)-A(T)))}{\beta+\exp(-\omega_{mn}^2A(T))}(C_{mn}(s)\cos m\theta+D_{mn}(s)\sin m\theta)J_m(\omega_{mn}r)\mathrm{d}s\Big] \tag{15}$$

where the parameter $\beta=\beta(\epsilon)>0$ can be chosen later.

## 2 Regularization and stability

For clarity, we denote that $\|\cdot\|$ is the norm in $L^2([0,r_0]\times[0,2\pi])$. We also assume the solution always exists.

### 2.1 Main results

**Theorem 2.1** (Stability of the modified method) Let $u^{\epsilon}(g)$ and $u^{\epsilon}(h)$ be defined by Eq. (15) corresponding to the final values $g$ and $h$ in $L^2([0,r_0]\times[0,2\pi])$, respectively. Then we get

$$\|u^{\epsilon}(g)(\cdot,t)-u^{\epsilon}(h)(\cdot,t)\|\leqslant \beta^{\frac{pt}{qT}-1}\|g-h\|,\ 0\leqslant t\leqslant T.$$

The error estimate between the exact solution of Eqs. (6)～(8) and the regularized solution is presented.

**Theorem 2.2** (The error estimate) Let $\epsilon\in(0,T)$, $g_{\epsilon},g\in L^2([0,r_0]\times[0,2\pi])$, and $u$ be the exact solution of Eqs. (6)～(8). If

$$\pi\int_0^{r_0}\sum_{m=0}^{\infty}\Big(\sum_{n=1}^{\infty}\exp(\omega_{mn}^2A(T))\,|\bar{v}_{mn}(r,t)|\Big)^2+\Big(\sum_{n=1}^{\infty}\exp(\omega_{mn}^2A(T))\,|\tilde{v}_{mn}(r,t)|\Big)^2\mathrm{d}r\leqslant B^2$$

holds for all $t\in[0,T]$ and $\|g_{\epsilon}-g\|\leqslant\epsilon$, we can get

$$\|u(g)(\cdot,t)-u^{\epsilon}(g_{\epsilon})(\cdot,t)\|\leqslant \epsilon^{\frac{1}{2}(\frac{pt}{qT}+1)}+\sqrt{\epsilon}B,\ 0\leqslant t\leqslant T.$$

### 2.2 Proofs of the main theorems

We need the following two lemmas to get

Theorems 2.1 and 2.2.

**Lemma 2.1** Let $\eta > 0$, $0 \leqslant a \leqslant b$, $b > 0$, then for all $k \in N$, one has

$$\frac{e^{ka}}{1+\eta e^{kb}} \leqslant \eta^{-\frac{a}{b}}.$$

**Proof**

$$\frac{e^{ka}}{1+\eta e^{kb}} = \frac{e^{ka}}{(1+\eta e^{kb})^{\frac{a}{b}}(1+\eta e^{kb})^{1-\frac{a}{b}}} \leqslant \frac{e^{ka}}{(1+\eta e^{kb})^{\frac{a}{b}}} \leqslant \eta^{-\frac{a}{b}}.$$

**Lemma 2.2** Let $0 < \beta < 1$, then for all $t \in [0, T]$, we have

$$\frac{\beta e^{-kA(t)}}{\beta + e^{-kA(T)}} \leqslant \beta^{\frac{pt}{qT}}.$$

**Proof** From Eq. (4), we have $pt \leqslant A(t) \leqslant qt$. According to Lemma 2.1 and the condition $0 < \beta < 1$, we obtain

$$\frac{\beta e^{-kA(t)}}{\beta + e^{-kA(T)}} = \beta \frac{e^{k(A(T)-A(t))}}{1+\beta e^{kA(T)}} \leqslant \beta \cdot \beta^{\frac{A(t)}{A(T)}-1} \leqslant \beta^{\frac{pt}{qT}}.$$

**Proof of Theorem 2.1**

From Eq. (15), we have

$$u^{\epsilon}(g)(r,\theta,t) = \sum_{m=0}^{\infty} \phi_m^{\epsilon}(g)(r,t)\cos m\theta + \psi_m^{\epsilon}(g)(r,t)\sin m\theta \tag{16}$$

$$u^{\epsilon}(h)(r,\theta,t) = \sum_{m=0}^{\infty} \phi_m^{\epsilon}(h)(r,t)\cos m\theta + \psi_m^{\epsilon}(h)(r,t)\sin m\theta \tag{17}$$

where

$$\phi_m^{\epsilon}(g)(r,t) = \sum_{n=1}^{\infty}\Big[\frac{\exp(-\omega_{mn}^2 A(t))}{\beta+\exp(-\omega_{mn}^2 A(T))}A_{mn}(g) - \int_t^T \frac{\exp(\omega_{mn}^2(A(s)-A(t)-A(T)))}{\beta+\exp(-\omega_{mn}^2 A(T))}C_{mn}(s)\mathrm{d}s\Big]J_m(\omega_{mn}r),$$

$$\psi_m^{\epsilon}(g)(r,t) = \sum_{n=1}^{\infty}\Big[\frac{\exp(-\omega_{mn}^2 A(t))}{\beta+\exp(-\omega_{mn}^2 A(T))}B_{mn}(g) - \int_t^T \frac{\exp(\omega_{mn}^2(A(s)-A(t)-A(T)))}{\beta+\exp(-\omega_{mn}^2 A(T))}D_{mn}(s)\mathrm{d}s\Big]J_m(\omega_{mn}r).$$

The $\phi_m^{\epsilon}(h)(r,t)$ and $\psi_m^{\epsilon}(h)(r,t)$ can be obtained similarly.

According to Lemma 2.2 and the definition of $A_{mn}$ and $B_{mn}$, we get

$$\begin{aligned}
&\| u^{\epsilon}(g)(\cdot,t) - u^{\epsilon}(h)(\cdot,t)\|^2 = \pi\sum_{m=0}^{\infty}\frac{2}{\delta_m} | \phi_m^{\epsilon}(g)(\cdot,t) - \phi_m^{\epsilon}(h)(\cdot,t)|^2 + \\
&\pi\sum_{m=1}^{\infty} | \psi_m^{\epsilon}(g)(\cdot,t) - \psi_m^{\epsilon}(h)(\cdot,t)|^2 = \pi\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{2}{\delta_m} \Big| \frac{\exp(-\omega_{mn}^2 A(t))}{\beta+\exp(-\omega_{mn}^2 A(T))}J_m(\omega_{mn}r)(A_{mn}(g)-A_{mn}(h))\Big|^2 + \\
&\pi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big| \frac{\exp(-\omega_{mn}^2 A(t))}{\beta+\exp(-\omega_{mn}^2 A(T))}J_m(\omega_{mn}r)(B_{mn}(g)-B_{mn}(h))\Big|^2 = \\
&\pi\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{2}{\delta_m}\Big| \frac{\exp(\omega_{mn}^2(A(T)-A(t)))}{1+\beta\exp(\omega_{mn}^2 A(T))}J_m(\omega_{mn}r)(A_{mn}(g)-A_{mn}(h))\Big|^2 + \\
&\pi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big| \frac{\exp(\omega_{mn}^2(A(T)-A(t)))}{1+\beta\exp(\omega_{mn}^2 A(T))}J_m(\omega_{mn}r)(B_{mn}(g)-B_{mn}(h))\Big|^2 \leqslant \\
&\beta^{2(\frac{A(t)}{A(T)}-1)} \| g-h\|^2
\end{aligned} \tag{18}$$

Therefore, we obtain

$$\| u^{\epsilon}(g)(\cdot,t) - u^{\epsilon}(h)(\cdot,t)\| \leqslant \beta^{\frac{A(t)}{A(T)}-1}\| g-h\| \leqslant \beta^{\frac{pt}{qT}-1}\| g-h\|.$$

The proof of Theorem 2.1 is complete.

**Proof of Theorem 2.2**

Considering the observation value $g_{\epsilon}$ with error $\epsilon$, and according to (18), we can obtain

$$\| u^{\epsilon}(g_{\epsilon})(\cdot,t) - u^{\epsilon}(g)(\cdot,t)\| \leqslant \beta^{\frac{A(t)}{A(T)}-1}\| g_{\epsilon}-g\| \leqslant \beta^{\frac{pt}{qT}-1}\cdot\epsilon \tag{19}$$

According to Eqs. (14) and (16), we have

$$u(r,\theta,t) - u^{\epsilon}(r,\theta,t) = \sum_{m=0}^{\infty}(\phi_m(r,t) - \phi_m^{\epsilon}(g)(r,t))\cos m\theta + (\psi_m(r,t) - \psi_m^{\epsilon}(g)(r,t))\sin m\theta =$$

$$\sum_{m=0}^{\infty}\Big(\cos m\theta\sum_{n=1}^{\infty}\frac{\beta}{\beta+\exp(-\omega_{mn}^2A(T))}\bar{v}_{mn}(r,t)+\sin m\theta\sum_{n=1}^{\infty}\frac{\beta}{\beta+\exp(-\omega_{mn}^2A(T))}\tilde{v}_{mn}(r,t)\Big).$$

Then

$$\|u^\epsilon(\cdot,t)-u(\cdot,t)\|^2=\int_0^{2\pi}\int_0^{r_0}|u(r,\theta,t)-u^\epsilon(r,\theta,t)|^2\mathrm{d}r\mathrm{d}\theta\leqslant$$
$$\pi\int_0^{r_0}\sum_{m=0}^{\infty}\Big[\Big(\sum_{n=1}^{\infty}\frac{\beta}{\beta+\exp(-\omega_{mn}^2A(T))}\bar{v}_{mn}(r,t)\Big)^2+\Big(\sum_{n=1}^{\infty}\frac{\beta}{\beta+\exp(-\omega_{mn}^2A(T))}\tilde{v}_{mn}(r,t)\Big)^2\Big]\mathrm{d}r\leqslant$$
$$\beta^2\pi\int_0^{r_0}\sum_{m=0}^{\infty}\Big(\sum_{n=1}^{\infty}\exp(\omega_{mn}^2A(T))\,|\bar{v}_{mn}(r,t)|\Big)^2+\Big(\sum_{n=1}^{\infty}\exp(\omega_{mn}^2A(T))\,|\tilde{v}_{mn}(r,t)|\Big)^2\mathrm{d}r.$$

Consequently, according to the hypothesis, we get

$$\|u^\epsilon(\cdot,t)-u(\cdot,t)\|\leqslant\beta B \tag{20}$$

Hence, we have

$$\|u^\epsilon(g_\epsilon)(\cdot,t)-u(g)(\cdot,t)\|\leqslant\|u^\epsilon(g_\epsilon)(\cdot,t)-u^\epsilon(g)(\cdot,t)\|+\|u^\epsilon(g)(\cdot,t)-u(g)(\cdot,t)\|\leqslant\beta^{\frac{pt}{qT}-1}\cdot\epsilon+\beta B \tag{21}$$

If we take $\beta=\sqrt{\epsilon}$, we get the proof of Theorem 2.2.

Therefore, under the condition of the prior estimate of the solution, we obtain the stable estimate of the solution. In particular, at the time of $t=0$, we get

$$\|u^\epsilon(g_\epsilon)(\cdot,0)-u(\cdot,0)\|\leqslant\frac{\epsilon}{\beta}+\beta B \tag{22}$$

We take the regularization parameter $\beta=\sqrt{\epsilon B}$ to obtain the optimal approximation error

$$\|u^\epsilon(g_\epsilon)(\cdot,0)-u(\cdot,0)\|\leqslant2\sqrt{\epsilon B} \tag{23}$$

## 3 Numerical experiment

Consider Eqs. (1)～(3) with $T=1$ and $r_0=1$, and

$$\left.\begin{aligned}&a(t)=2t+1,\\&f(x,y,t)=\frac{(1+2t)(x^2+y^2+3)}{\exp(t^2+t)}\end{aligned}\right\} \tag{24}$$

If

$$g(x,y)=u(x,y,1)=\frac{1-(x^2+y^2)}{\mathrm{e}^2}=\frac{1-r^2}{\mathrm{e}^2} \tag{25}$$

we can obtain the unique solution

$$u(x,y,t)=\frac{1-(x^2+y^2)}{\exp(t^2+t)}.$$

Then we have $u(x,y,0)=1-(x^2+y^2)$.

Consider the measured data

$$g_\epsilon(r,\theta)=(1+\sqrt{6}\epsilon\mathrm{e}^2)\,g(r,\theta) \tag{26}$$

then we have

$$\|g_\epsilon-g\|=\sqrt{6}\epsilon\mathrm{e}^2\|g\|=\epsilon \tag{27}$$

From Eqs. (24) and (25), we can find that the function considered is independent of $\theta$, so we can get $m=0$. Therefore, we have the regularized solution for the case $t=0$ from Eqs. (15) and (26).

$$u^\epsilon(r,\theta,0)=\sum_{n=1}^{\infty}\Big[\frac{1}{\beta+\exp(-2\omega_{0n}^2)}A_{0n}^\epsilon J_0(\omega_{0n}r)-\int_0^1\frac{\exp(\omega_{0n}^2(s^2+s-2))}{\beta+\exp(-2\omega_{0n}^2)}C_{0n}(s)J_0(\omega_{0n}r)\mathrm{d}s\Big] \tag{28}$$

where

$$A_{0n}^\epsilon=\frac{2}{\pi J_1^2(\omega_{0n})}\int_0^1\int_0^{2\pi}g_\epsilon(r,\theta)\,J_0(\omega_{0n}r)r\mathrm{d}r\mathrm{d}\theta,$$

$$C_{0n}(s)=\frac{2}{\pi J_1^2(\omega_{0n})}\int_0^1\int_0^{2\pi}f(r,\theta,s)\,J_0(\omega_{0n}r)r\mathrm{d}r\mathrm{d}\theta.$$

When we use the numerical method to invert the value of the moment $t=0$, we need to truncate the series. Let $B_n$ represent the partial sum of series $B$ and $\beta=\sqrt{\epsilon B_n}$, we can see that

$$\|u(g)(\cdot,0)-u_n^\epsilon(g_\epsilon)(\cdot,0)\|\leqslant\sqrt{\epsilon B_n}.$$

From the above inequalities, we can see that the accuracy $\|u(g)(\cdot,0)-u_n^\epsilon(g_\epsilon)(\cdot,0)\|$ is not only related to $\epsilon$, but also to the truncation degree $n$. We also show this phenomenon in the following figures. We take $n=6$ on Fig. 1, that is the sum of the first six terms of a series in Eq. (28). From Fig. 1 (a), when $\epsilon=0$, which corresponds the case where no regularization parameter is introduced into Eq. (10), we can see that the backward problem is seriously ill-posed. In Fig. 1(b), we take the values of $\epsilon$ as $1\times10^{-5}$, $1\times10^{-6}$, $1\times10^{-7}$, respectively, and we can find the effectiveness of the proposed method.

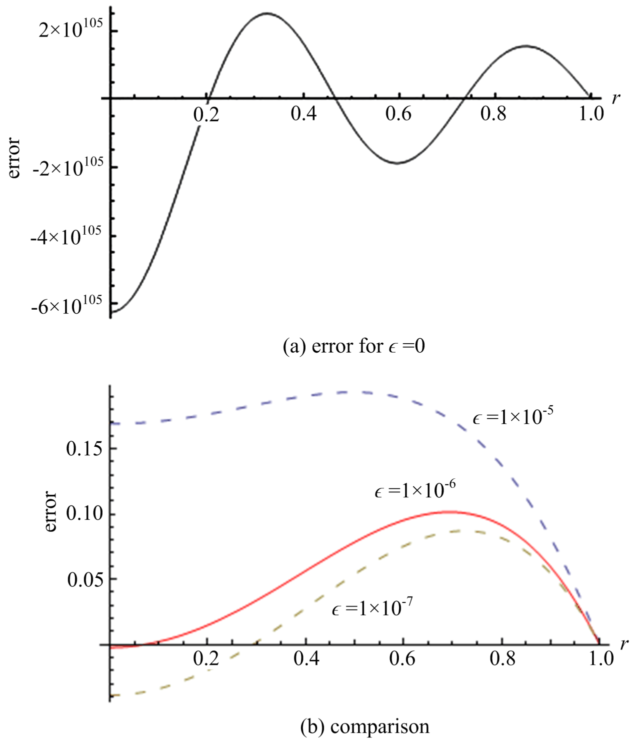


**Fig. 1 Comparison of errors in different values of $\epsilon$**

## 4 Conclusion

In this paper, we discuss the backward problem of parabolic equations with time-independent coefficients over two-dimensional circular domains. In order to solve the problem by separating variables, we transform the circular domain into a rectangular region by parameter transformation, but the following problem is that we need to introduce the Bessel function, which makes the problem difficult. However, with the help of a priori estimate, we still get the regularized solution. By choosing appropriate regularization parameters, we obtain relatively satisfactory results. A numerical example verifies the effectiveness of the presented method.

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