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A special class of near-perfect numbers

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Abstract: Let $\alpha \geqslant 2$ be an integer, p_1 and p_2 be odd prime numbers with $p_1 < p_2$. By using elementary methods and techniques, it was proved that there are no near-perfect numbers of the form $2^{\alpha-1} p_1^2 p_2^2$ with the redundant divisor $d \in \{1, p_1^2, p_2^2, p_1 p_2, p_1 p_2^2, p_1^2 p_2\}$, and then an equivalent condition for near-perfect numbers of the form $2^{\alpha-1} p_1^2 p_2^2$ with the redundant divisor $d \in \{p_1, p_2\}$ was obtained. Furthermore, for a fixed positive integer $k \geqslant 2$, by generalizing the definition of near-perfect numbers to be k-weakly-near-perfect numbers, it was proved that there are no k-weakly-near-perfect numbers of the form $n = 2^{\alpha-1} p_1^2 p_2^2$ when $k \geqslant 3$.

Key words: perfect number; near-perfect number; redundant divisor; k-weakly-near-perfect number

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一类特殊的 near-perfect 数

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摘要:设正整数 $\alpha \geqslant 2$, p_1, p_2 为奇质数且 $p_1 < p_2$.利用初等的方法和技巧,证明了不存在形如 $2^{\alpha-1} p_1^2 p_2^2$ 的以 $d \in \{1, p_1^2, p_2^2, p_1 p_2, p_1 p_2^2, p_1^2 p_2\}$ 为冗余因子的 near-perfect 数,并给出存在形如 $2^{\alpha-1} p_1^2 p_2^2$ 的以 $d \in \{p_1, p_2\}$ 为冗余因子的 near-perfect 数的一个等价刻画.进而,给定正整数 $k \geqslant 2$,通过推广 near-perfect 数的定义至 k 弱 near-perfect 数,证明了当 $k \geqslant 3$ 时,不存在形如 $2^{\alpha-1} p_1^2 p_2^2$ 的以 $d \in \{p_1^2, p_2^2\}$ 为冗余因子的 k 弱 near-perfect 数.

关键词:perfect 数;near-perfect 数;冗余因子;k 弱 near-perfect 数

0 Introduction

Definition 0.1[1] Let n be a positive integer, $D = \{d : d \mid n, 1 \leqslant d \leqslant n\}$ and $\sigma(n) = \sum_{d \in D} d$.

① If $\sigma(n) = 2n$, then n is called a perfect number.

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② If there exists some $S \subseteq D - \{n\}$ such that $n = \sum_{d \in S} d$, then n is called a pseudoperfect number.

③ If there exists some $d \in D - \{n\}$ such that $\sigma(n) = 2n + d$, then n is called a near-perfect number with the redundant divisor d.

As early as 300 BC, Euclid gave a sufficient condition for perfect numbers in his great work *Euclid's Elements* as follows.

Proposition 0.1 If p and $2^p - 1$ are both primes, then $2^{p-1}(2^p - 1)$ is a perfect number.

In 1747, Euler proved that the above condition is also necessary for any even perfect number. Since the prime of the form $2^p - 1$ is the well-known Mersenne prime, so the number of Mersenne primes depends on the number of even perfect numbers. It is still an open problem whether there are infinite Mersenne primes. So far, all known perfect numbers are even numbers, which naturally lead to the following.

Question Whether there is an odd perfect number?

Euler gave a necessary condition for odd perfect numbers as follows.

Proposition 0.2 If n is an odd perfect number, then $n = p^\alpha \prod_i^s q_i^{2\beta_i}$, where $p \equiv \alpha \equiv 1 \pmod 4$, p, q_i are distinct odd primes, and β_i are positive integers ($i = 1, 2, \cdots, s$).

In the last few decades, many problems on odd perfect numbers, such as determining the number of distinct prime factors[7] or the lower bound of an odd perfect number[6], have been studied. But the existence of an odd perfect number is still open, which makes people discuss near-perfect numbers closely related to perfect numbers. In recent years, some good results have been obtained. For example, in 2012, Pollack and Shevelev gave 3 classes of even near-perfect numbers which are all in the form $2^\alpha p^\beta$, where α, β are both integers, and p is an odd prime[9]. And then Ren and Chen completely determined all near-perfect numbers with two prime factors[10]. In 2015, Li and Liao gave an equivalent condition for near-perfect numbers of the form $2^\alpha p^\beta$ or $2^\alpha p_1 p_2$, where α and β are both positive integers, p, p_1, p_2 are odd primes and $p_1 \neq p_2$[1]. Recently, Li, et al. discussed near-perfect numbers of the form $n = 2^{\alpha-1} p_1^{2k_1} p_2^{2k_2}$ and proved that there is no near-perfect numbers of the form $2^{\alpha-1} p_1^2 p_2^2$ with the redundant divisor $p_1^2 p_2^2$, where k_1 and k_2 are both positive integers[4].

On the other hand, it is easy to see that all distinct positive factors of $2^{\alpha-1} p_1^2 p_2^2$ form the set
$A = \{1, p_1, p_2, p_1 p_2, p_1^2, p_2^2, p_1^2 p_2, p_1 p_2^2, p_1^2 p_2^2\}$,
which also includes the possible redundant factors when $2^\alpha p_1 p_2$ is near-perfect. Based on this fact, the present paper continues to study the issue.

1 Main results

For any fixed positive integer k, by using elementary techniques and methods, the present paper generalizes the definition of the near-perfect number to be the k-weakly-near-perfect number and proves that there are no $k(\geqslant 3)$-weakly-near-perfect numbers of the form $n = 2^{\alpha-1} p_1^2 p_2^2$. We improve the corresponding results given by Refs. [1,4] and prove the following main results.

Definition 1.1 Let n and $k \geqslant 2$ be two positive integers, $D = \{d : d \mid n, 1 \leqslant d \leqslant n\}$ and $\sigma(n) = \sum_{d \in D} d$. If $\sigma(n) = kn + d$, then n is called a k-weakly-near-perfect number with the redundant divisor d. Obviously, any 2-weakly-near-perfect number is near-perfect.

Theorem 1.1 Let $\alpha \geqslant 2$ be an integer, p_1, p_2 be both odd primes with $p_1 < p_2$. Then there is no near-perfect numbers of the form $n = 2^{\alpha-1} p_1^2 p_2^2$ with the redundant divisor $d \in \{p_1 p_2, p_1 p_2^2, 1, p_1^2, p_2^2, p_1^2 p_2\}$.

Theorem 1.2 Let $\alpha \geqslant 2$ be an integer, p_1, p_2 be two odd primes with $p_1 < p_2$. Then $n = 2^{\alpha-1} p_1^2 p_2^2$ is a near-perfect number with the redundant divisor $d \in \{p_1, p_2\}$ if and only if $\dfrac{p_1^2 p_2^2}{d} = k_1 k_2 - k_3$, where

① if $d = p_1$, then

$$k_1 p_1 = p_2^2 + p_2 + 1,\ k_2 = p_1^2 + p_1 + 1,$$

$$k_3(2^\alpha-1)=p_1p_2^2+1 \qquad (1)$$

② if $d=p_2$, then

$$k_1=p_2^2+p_2+1,\ k_2p_2=p_1^2+p_1+1,$$
$$k_3(2^\alpha-1)=p_1^2p_2+1 \qquad (2)$$

Theorem 1.3 Let $\alpha\geqslant 2$, $k\geqslant 3$ be two integers, p_1, p_2 be both odd primes with $p_1<p_2$. Then there are no k-weakly-near-perfect numbers of the form $n=2^{\alpha-1}p_1^2p_2^2$.

2 Proofs of the main results

For convenience, throughout this section, $\alpha\geqslant 2$ is an integer, p_1 and p_2 are both odd primes with $p_1<p_2$. Before proving our main results, the following three lemmas are needed.

Lemma 2.1[4] Suppose that $n=2^{\alpha-1}p_1^2p_2^2$ is a near-perfect number with the redundant divisor d.

① If $2^\alpha-1$ is a prime number, then $\gcd(p_1p_2, 2^\alpha-1)=1$.

② If $2^\alpha-1$ is a composite number and $d\neq p_1^2p_2^2$, then $\gcd(p_1p_2, 2^\alpha-1)=1$.

Lemma 2.2[4] Suppose that $n=2^{\alpha-1}p_1^{n_1}p_2^{n_2}$ ($\alpha\geqslant 2, p_1<p_2, n_1, n_2\in 2\mathbb{N}^+$) is a near-perfect number with the redundant divisor $d=p_1^{k_1}p_2^{k_2}$, where k_1 and k_2 are both positive integers and $(k_1,k_2)\neq(n_1,n_2)$. Then $p_i^{k_i}\mid\sigma(p_j^{n_j})$ with $1\leqslant i\neq j\leqslant 2$, $1\leqslant k_i\leqslant n_i (i=1,2)$.

Lemma 2.3[2] Let a and b be two positive integers with $a<b$. If $a\mid b^2+b+1$ and $b\mid a^2+a+1$, then $3a\leqslant b<5a$.

Proof for Theorem 1.1

Suppose that $n=2^{\alpha-1}p_1^2p_2^2$ is a near-perfect number with the redundant divisor $d\in\{p_1p_2, p_1p_2^2, 1, p_1^2, p_2^2, p_1^2p_2\}$, then $\sigma(n)=2n+d$, i.e.,

$$(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+d \qquad (3)$$

(Ⅰ) For $d=p_1p_2$, from (3) we have

$$(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+p_1p_2 \qquad (4)$$

Note that p_1 and p_2 are both odd primes, namely, $p_1\equiv p_2\equiv 1,3 \pmod 4$. Hence there are four cases as the following.

Case 1 If $p_1\equiv p_2\equiv 1\pmod 4$, then from (4) we can get

$$3\equiv(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+p_1p_2\equiv 1\pmod 4,$$

which is a contradiction.

Case 2 If $p_1\equiv p_2\equiv 3\pmod 4$, then from (4) we have

$$3\equiv(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+p_1p_2\equiv 1\pmod 4,$$

which is impossible.

Case 3 If $p_1\equiv 1\pmod 4$ and $p_2\equiv 3\pmod 4$, then from (4) we know that

$$1\equiv(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+p_1p_2\equiv 3\pmod 4,$$

which is a contradiction.

Case 4 If $p_1\equiv 3\pmod 4$ and $p_2\equiv 1\pmod 4$, then from (4) we can get

$$1\equiv(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+p_1p_2\equiv 3\pmod 4,$$

which is also a contradiction.

(Ⅱ) For $d=p_1p_2^2$, from (3) we have

$$(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+p_1p_2^2.$$

Thus from Lemma 2.2, we can get $p_2^2\mid p_1^2+p_1+1$. Note that p_1 and p_2 are both odd primes and $p_1<p_2$, hence $(p_1+1)^2<p_2^2$, thus

$$(p_1+1)^2<p_2^2<p_1^2+p_1+1<(p_1+1)^2,$$

which is impossible.

(Ⅲ) For $d=1$, from (3) we can get

$$(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+1,$$

and then $(2^\alpha-1)\mid 2^\alpha p_1^2p_2^2+1$, i.e., $(2^\alpha-1)\mid p_1^2p_2^2+1$, which means that

$$p_1^2p_2^2\equiv(p_1p_2)^2\equiv -1\pmod{2^\alpha-1} \qquad (5)$$

Note that $\alpha\geqslant 2$, thus $2^\alpha-1\equiv 3\pmod 4$, and so $\left(\frac{-1}{2^\alpha-1}\right)=-1$, where $(\ast)$ is the Jacobi symbol. This is a contradiction with the identity (5).

(Ⅳ) For $d=p_1^2$, from (3) we can get

$$(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+p_1^2.$$

While from Lemmas 2.1 and 2.2 and $d=p_1^2$ we know that $\gcd(p_1, (2^\alpha-1)(p_1^2+p_1+1))=1$, and so

$$(2^\alpha-1)(p_1^2+p_1+1)\left(\frac{p_2^2+p_2+1}{p_1^2}\right)=2^\alpha p_2^2+1,$$

which means that $(2^\alpha-1)\mid 2^\alpha p_2^2+1$, equivalently, $(2^\alpha-1)\mid p_2^2+1$, i.e., $p_2^2\equiv -1 \pmod{2^\alpha-1}$. By the same proof of (Ⅲ), we can get a contradiction.

(Ⅴ) For $d=p_2^2$, by the same proof of (Ⅳ), we can also get a contradiction.

(Ⅵ) If $n=2^{\alpha-1}p_1^2p_2^2$ is a near-perfect number with the redundant divisor $d=p_1^2p_2$, then from (3) we know that

$$(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+p_1^2p_2 \tag{6}$$

Thus by Lemmas 2.1 and 2.2 we have $p_1^2\mid p_2^2+p_2+1$ and $p_2\mid p_1^2+p_1+1$. This means that there are some positive integers k_1 and k_2 such that

$$k_1p_1^2=p_2^2+p_2+1,\ k_2p_2=p_1^2+p_1+1.$$

Now from Lemma 2.3 we have $3p_1<p_2<5p_1$, and so

$$k_1p_1^2=p_2^2+p_2+1<25p_1^2+5p_1+1<27p_1^2,$$
$$k_1p_1^2=p_2^2+p_2+1>9p_1^2,$$

hence $10\leqslant k_1\leqslant 26$. Note that $k_1k_2p_2=k_1p_1^2+k_1p_1+k_1=p_2^2+p_2+1+k_1p_1+k_1$, thus

$$(k_1k_2-p_2-1)p_2=1+k_1p_1+k_1,$$

and so $k_1p_1\equiv -1-k_1 \pmod{p_2}$. Therefore

$$k_1^2k_2p_2=k_1^2p_1^2+k_1^2p_1+k_1^2\equiv(-1-k_1)^2+k_1(-1-k_1)+k_1^2=k_1^2+k_1+1 \pmod{p_2},$$

i.e., $p_2\mid k_1^2+k_1+1$.

Case 1 From $k_1=10$ we have $k_1^2+k_1+1=111=3\times 37$. While $p_2>p_1\geqslant 3$, and so $p_2=37$. Thus from $k_1p_1^2=p_2^2+p_2+1$ we can get $10p_1^2=37^2+37+1=1407$. This is impossible.

Case 2 From $k_1=11$ we have $k_1^2+k_1+1=133=7\times 19$. While $p_2>p_1\geqslant 3$, and so $p_2=7$ or 19. Form $p_2=7$ and $k_1p_1^2=p_2^2+p_2+1$, we know that $11p_1^2=7^2+7+1=57$, which is impossible. Hence $p_2=19$, similarly we can also get a contradiction.

For other cases, namely, $k=12,13,\cdots,26$, in the same way as for cases 1 and 2, we also have a contradiction.

Now from (Ⅰ)~(Ⅵ), we complete the proof of Theorem 1.1.

Proof for Theorem 1.2

(Ⅰ) If $n=2^{\alpha-1}p_1^2p_2^2$ is a near-perfect number with the redundant divisor $d=p_1$, then from (3) we know that

$$(2^\alpha-1)(p_1^2+p_1+1)(p_2^2+p_2+1)=2^\alpha p_1^2p_2^2+p_1 \tag{7}$$

Thus by Lemmas 2.1 and 2.2 we have $p_1\mid p_2^2+p_2+1$ and $(2^\alpha-1)\mid p_1p_2^2+1$. This means that there are some positive integers k_1, k_2 and k_3 such that

$$k_1p_1=p_2^2+p_2+1,\ k_2=p_1^2+p_1+1,$$

and

$$k_3(2^\alpha-1)=p_1p_2^2+1.$$

Therefore

$$k_1k_2-k_3=\frac{p_2^2+p_2+1}{p_1}\cdot(p_1^2+p_1+1)-\frac{p_1p_2^2+1}{2^\alpha-1}=$$
$$\frac{1}{(2^\alpha-1)p_1}\cdot(2^\alpha-1)\cdot p_1\cdot\left(\frac{p_2^2+p_2+1}{p_1}\cdot(p_1^2+p_1+1)-\frac{p_1p_2^2+1}{2^\alpha-1}\right)=$$
$$\frac{1}{(2^\alpha-1)p_1}[(2^\alpha-1)(p_1^2+p_1+1)\cdot(p_2^2+p_2+1)-p_1^2p_2^2-p_1],$$

thus by (7) we can obtain

$$k_1k_2-k_3=\frac{1}{(2^\alpha-1)p_1}(2^\alpha p_1^2p_2^2-p_1^2p_2^2)=p_1p_2^2=\frac{p_1^2p_2^2}{d},$$

which means that (1) is true.

On the other hand, if (1) is true, then we have

$$\sigma(n)-2n=(2^\alpha-1)(p_1^2+p_1+1)\cdot(p_2^2+p_2+1)-2^\alpha p_1^2p_2^2=$$
$$(2^\alpha-1)k_1k_2p_1-2^\alpha p_1^2p_2^2=$$
$$p_1[(2^\alpha-1)k_1k_2-(2^\alpha-1)p_1p_2^2-p_1p_2^2]=$$
$$p_1[(2^\alpha-1)(k_1k_2-p_1p_2^2)-p_1p_2^2]=$$
$$p_1[(2^\alpha-1)k_3-p_1p_2^2]=p_1=d,$$

thus from Definition 0.1, $n=2^{\alpha-1}p_1^2p_2^2$ is a near-perfect number with the redundant divisor $d=p_1$.

Thus we complete the proof of (Ⅰ).

(Ⅱ) If $n=2^{\alpha-1}p_1^2p_2^2$ is a near-perfect number with the redundant divisor $d=p_2$, by the same proof of (Ⅰ), we can get (Ⅱ).

From the above we complete the proof of

Theorem 1.2.

Proof for Theorem 1.3

Suppose that $\alpha \geqslant 2$, $k \geqslant 3$ are both positive integers and $n = 2^{\alpha-1} p_1^2 p_2^2$ is a k-weakly-near-perfect number with the redundant divisor d, where p_1, p_2 are both odd primes and $p_1 < p_2$. Then $\sigma(n) = kn + d$, namely,

$$\frac{\sigma(n)}{n} = k + \frac{d}{n} \tag{8}$$

thus we have

$$(2 - \frac{1}{2^{\alpha-1}})(1 + \frac{p_1+1}{p_1^2})(1 + \frac{p_2+1}{p_2^2}) = k + \frac{d}{2^{\alpha-1} p_1^2 p_2^2} \tag{9}$$

Note that the left side of (9) reaches the maximum value if and only if $p_1 = 3$ and $p_2 = 5$. Equivalently, for $\alpha \geqslant 2$ we have

$$(2 - \frac{1}{2^{\alpha-1}})(1 + \frac{p_1+1}{p_1^2})(1 + \frac{p_2+1}{p_2^2}) < 2 \times \frac{13}{9} \times \frac{31}{25} < 4 \tag{10}$$

Now from $0 < \frac{d}{n} < 1$ and (9)~(10) we know that $k = 2$ or 3, and so $k = 3$ from the assumption that $k \geqslant 3$. In this case, if $p_1 > 3$, then from (10) we have

$$(2 - \frac{1}{2^{\alpha-1}})(1 + \frac{p_1+1}{p_1^2})(1 + \frac{p_2+1}{p_2^2}) < 2 \times \frac{31}{25} \times \frac{57}{49} = 2.88\cdots \tag{11}$$

which is a contradiction with (9). Hence $p_1 = 3$. On the other hand, if $p_2 \geqslant 29$, then from (10) we have

$$(2 - \frac{1}{2^{\alpha-1}})(1 + \frac{p_1+1}{p_1^2})(1 + \frac{p_2+1}{p_2^2}) < 2 \times \frac{13}{9} \times \frac{1+29+29^2}{29^2} = 2.9919\cdots, \tag{12}$$

which is also a contradiction with (9). Hence $p_2 \leqslant 23$, namely, $p_2 \in \{5, 7, 11, 13, 17, 19, 23\}$.

Note that $n = 2^{\alpha-1} p_1^2 p_2^2$ and $\sigma(n) = kn + d$ with the odd redundant divisor d. Hence $d \mid p_1^2 p_2^2$, thus we have the following 7 cases.

Case 1 If $p_1 = 3$ and $p_2 = 7$, then

$$(2^\alpha - 1)(1 + 3 + 3^2)(1 + 7 + 7^2) = 2^{\alpha-1} 3^3 7^2 + d,$$

thus $159 \cdot 2^{\alpha-1} = 741 + d$, and so $\alpha \geqslant 4$. Therefore $741 + d = 159 \cdot 2^{\alpha-1} \geqslant 1272$, namely, $d \geqslant 531 > 3^2 7^2$, which a contradiction.

Case 2 If $p_1 = 3$ and $p_2 = 11$, then

$$(2^\alpha - 1)(1 + 3 + 3^2)(1 + 11 + 11^2) = 2^{\alpha-1} 3^3 11^2 + d,$$

thus $191 \cdot 2^{\alpha-1} = 1729 + d$, and so $\alpha \geqslant 5$. Therefore $1729 + d = 191 \cdot 2^{\alpha-1} \geqslant 3056$, namely, $d \geqslant 1327 > 3^2 11^2$, which a contradiction.

Case 3 If $p_1 = 3$ and $p_2 = 13$, then

$$(2^\alpha - 1)(1 + 3 + 3^2)(1 + 13 + 13^2) = 2^{\alpha-1} 3^3 13^2 + d,$$

thus $195 \cdot 2^{\alpha-1} = 2379 + d$, and so $\alpha \geqslant 5$. For $\alpha = 5$ we have $2379 + d = 195 \cdot 2^{\alpha-1} = 3120$, namely, $d = 741 = 3 \times 13 \times 19$, this is a contradiction. Hence $\alpha \geqslant 6$, in this case, $2379 + d = 195 \cdot 2^{\alpha-1} \geqslant 6240$, namely, $d \geqslant 3861 > 3^2 13^2$, which also a contradiction.

Case 4 If $p_1 = 3$ and $p_2 = 17$, then

$$(2^\alpha - 1)(1 + 3 + 3^2)(1 + 17 + 17^2) = 2^{\alpha-1} 3^3 17^2 + d,$$

and so $179 \cdot 2^{\alpha-1} = 3991 + d$, hence $\alpha \geqslant 6$. For $\alpha = 6$ we have $3991 + d = 179 \cdot 2^{\alpha-1} = 5728$, namely, $d = 1737 = 3^2 \times 193$, which a contradiction. Hence $\alpha \geqslant 7$, in this case, $3991 + d = 179 \cdot 2^{\alpha-1} \geqslant 11456$, thus $d \geqslant 7465 > 3^2 17^2$, which also a contradiction.

Case 5 If $p_1 = 3$ and $p_2 = 19$, then

$$(2^\alpha - 1)(1 + 3 + 3^2)(1 + 19 + 19^2) = 2^{\alpha-1} 3^3 19^2 + d,$$

thus $159 \cdot 2^{\alpha-1} = 4953 + d$, and so $\alpha \geqslant 6$. For $\alpha = 6$ we have $4953 + d = 159 \cdot 2^{\alpha-1} = 5088$, namely, $d = 135 = 3^3 \times 5$, which a contradiction. Therefore $\alpha \geqslant 7$, thus we have $4953 + d = 159 \cdot 2^{\alpha-1} \geqslant 10176$, namely, $d \geqslant 5223 > 3^2 19^2$, which is a contradiction.

Case 6 If $p_1 = 3$ and $p_2 = 23$, then

$$(2^\alpha - 1)(1 + 3 + 3^2)(1 + 23 + 23^2) = 2^{\alpha-1} 3^3 23^2 + d,$$

thus $95 \cdot 2^{\alpha-1} = 7189 + d$, and so $\alpha \geqslant 8$. Therefore $7189 + d = 95 \cdot 2^{\alpha-1} \geqslant 12160$, namely, $d \geqslant 4971 > 3^2 23^2$, which a contradiction.

Case 7 If $p_1 = 3$ and $p_2 = 5$, then

$$(2^\alpha - 1)(1 + 3 + 3^2)(1 + 5 + 5^2) = 2^{\alpha-1} 3^3 5^2 + d,$$

and so $131 \cdot 2^{\alpha-1} = 403 + d$. Note that $1 \leqslant d \leqslant 3^2 5^2 = 225$, hence $404 \leqslant 131 \cdot 2^{\alpha-1} = 403 + d \leqslant 628$, and so $\alpha = 3$. Therefore $d = 131 \cdot 2^{\alpha-1} - 403 = 121 = 11^2$, which a contradiction.

Thus we complete the proof of Theorem 1.3.

Remark The present paper mainly discusses near-perfect numbers of the form $2^{\alpha-1}p_1^2p_2^2$. The issue is an extension of some classical number theory problems, such as perfect numbers or odd perfect numbers, which are not easy. The main related studies can be seen in Refs.[3, 5, 8, 11, 12].

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(上接第 898 页)

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