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n-color 1-2-3 compositions of positive integers

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Abstract: An n-color 1-2-3 composition of positive integers is defined as an n-color composition with only parts of size 1, 2 or 3. An n-color 1-2-3 palindromic composition is an n-color 1-2-3 composition that reads the same forward as backward. Here the generating function, explicit formulas and recurrence relations for the number of n-color 1-2-3 compositions and the n-color 1-2-3 palindromic compositions of positive integers were obtained. In addition, a relation between the number of 1-2-3 compositions of a positive integer and the number of compositions of a positive integer with parts omitting all multiples of size 3 was given. Furthermore, the generalized relation was obtained.

Key words: compositions of positive integers; n-color 1-2-3 composition; n-color 1-2-3 palindromic composition; generating function; explicit formula; recurrence relation; combinatorial proof

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正整数 n-color 1-2-3 有序分拆

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摘要:正整数的 n-color 1-2-3 有序分拆是指正整数的只含分部量是 1, 2 或者 3 的 n-color 有序分拆,而正整数的回文的 n-color 1-2-3 有序分拆是指只含有分部量是 1, 2 或者 3 的 n-color 有序分拆且分部量从左往右读与从右往左读是相等的. 给出了正整数的 n-color 1-2-3 有序分拆数和回文的 n-color 1-2-3 有序分拆数的生成函数、显式公式以及递推公式.还给出了正整数的 1-2-3 有序分拆数和正整数不含分部量是 3 的倍数的有序分拆数之间的一个关系式以及推广形式.

关键词:正整数的有序分拆;n-color 1-2-3 有序分拆;回文的 n-color 1-2-3 有序分拆;生成函数;显式公式;递推公式;组合证明

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0 Introduction

In the theory of compositions studying restricted parts of compositions has always been an interesting topic. In recent years, there has been a large number of researchers and many results in n-color compositions[1-9]. An n-color composition defined as a composition of a positive integer in which a part of size n may be assigned one of n colors[1]. As a brief example, there are 8 n-color compositions of 3. Viz., 3_1, 3_2, 3_3, 2_1+1_1, 2_2+1_1, 1_1+2_1, 1_1+2_2, $1_1+1_1+1_1$. A palindromic composition or palindrome[10] also referred to as a self-inverse composition[5,7] is a composition whose part sequence is the same whether it is read from left to right or right to left. For example, there are four palindromic compositions of 4. Viz., 4, 2+2, 1+2+1, 1+1+1+1. And there are nine n-color palindromic compositions of 4. Viz., 4_1, 4_2, 4_3, 4_4, 2_1+2_1, 2_2+2_2, $1_1+2_1+1_1$, $1_1+2_2+1_1$, $1_1+1_1+1_1+1_1$.

Ref.[10] gave the compositions having parts of size 1, 2 and 3 of positive integers, we refer to here as 1-2-3 compositions. Thus, for example, there are seven 1-2-3 compositions of 4. Viz., 1+3, 3+1, 2+2, 1+2+1, 2+1+1, 1+1+2, 1+1+1+1. Let $C_{1\text{-}2\text{-}3}(\nu)$ denote the number of 1-2-3 compositions of ν. Ref.[10] gave the generating function of the number of 1-2-3 compositions as $\frac{1}{1-x-x^2-x^3}=\sum_{n=0}^{\infty}T_{n+1}x^n$, where T_n is the n-th Tribonacci number: $T_1=1$, $T_2=1$, $T_3=2$, $T_n=T_{n-1}+T_{n-2}+T_{n-3}$ when $n>2$. i.e., $C_{1\text{-}2\text{-}3}(\nu)=T_{\nu+1}$.

In this paper, we will study n-color 1-2-3 compositions, n-color 1-2-3 palindromic compositions. The generating function, explicit formulas and recurrence relations for the number of n-color 1-2-3 compositions are presented in Section 1. In Section 2, we obtain the generating function and recurrence relations for the number of n-color 1-2-3 palindromic compositions. In Section 3, a relation between the number of 1-2-3 compositions of a positive ν and the number of compositions of ν with parts omitting all multiples of size 3 is given. Furthermore, the generalized relation is given.

In this paper, we denote the set of nonnegative integers by $Z^{\geqslant 0}$, the set of positive integers by $Z^{>0}$.

1 n-color 1-2-3 compositions

In this section, we first give the following definition.

Definition 1.1 An n-color 1-2-3 composition is an n-color composition with only parts of size 1, 2 and 3.

For example, the n-color 1-2-3 compositions of 3 are as follows.

$$3_1,\ 3_2,\ 3_3,\ 2_1+1_1,\ 2_2+1_1,$$
$$1_1+2_1,\ 1_1+2_2,\ 1_1+1_1+1_1.$$

We denote the number of n-color 1-2-3 compositions of ν by $A_{1\text{-}2\text{-}3}(\nu)$, and the number of n-color 1-2-3 compositions of ν into m parts by $A_{1\text{-}2\text{-}3}(m,\nu)$, respectively. We prove the following theorem.

Theorem 1.1 Let $A_{1\text{-}2\text{-}3}(m,q)$ and $A_{1\text{-}2\text{-}3}(q)$ denote the generating functions for $A_{1\text{-}2\text{-}3}(m,\nu)$ and $A_{1\text{-}2\text{-}3}(\nu)$, respectively. Then

$$A_{1\text{-}2\text{-}3}(m,q)=(q+2q^2+3q^3)^m \tag{1}$$

$$A_{1\text{-}2\text{-}3}(q)=\frac{q+2q^2+3q^3}{1-q-2q^2-3q^3} \tag{2}$$

$$A_{1\text{-}2\text{-}3}(m,\nu)=\sum_{2i-j=\nu-m}\binom{m}{i}\binom{i}{j}2^j\times 3^{i-j} \tag{3}$$

where $j\leqslant i\leqslant m$; i, $j\in Z^{\geqslant 0}$.

Proof Following Agarwal's proof in Ref.[1], we have

$$A_{1\text{-}2\text{-}3}(m,q)=\sum_{\nu=1}^{\infty}A_{1\text{-}2\text{-}3}(m,\nu)q^\nu=(q+2q^2+3q^3)^m,$$

this proves (1).

And

$$A_{1\text{-}2\text{-}3}(q)=\sum_{m=1}^{\infty}A_{1\text{-}2\text{-}3}(m,q)=\sum_{m=1}^{\infty}(q+2q^2+3q^3)^m=\frac{q+2q^2+3q^3}{1-q-2q^2-3q^3},$$

which proves (2).

Equating the coefficients of q^{ν} in (1), we have

$$A_{1\text{-}2\text{-}3}(m,\nu)=\sum_{2i-j=\nu-m}\binom{m}{i}\binom{i}{j}2^{j}\times 3^{i-j},$$

where $j\leqslant i\leqslant m$; i, $j\in Z^{\geqslant 0}$. This proves (3). Thus we complete the proof.

It is naturally to obtain the recurrence formula of the number of n-color 1-2-3 compositions from the generating function.

Theorem 1.2 Let $A_{1\text{-}2\text{-}3}(\nu)$ denote the number of n-color 1-2-3 compositions of ν. Then

$$A_{1\text{-}2\text{-}3}(1)=1,\ A_{1\text{-}2\text{-}3}(2)=3,\ A_{1\text{-}2\text{-}3}(3)=8,$$
$$A_{1\text{-}2\text{-}3}(\nu)=A_{1\text{-}2\text{-}3}(\nu-1)+2A_{1\text{-}2\text{-}3}(\nu-2)+3A_{1\text{-}2\text{-}3}(\nu-3)\ \text{for}\ \nu>3.$$

We now present the combinatorial proof although this recurrence relation is the straight result of Eq. (2) in Theorem 1.1.

Proof (Combinatorial) To prove that

$$A_{1\text{-}2\text{-}3}(\nu)=A_{1\text{-}2\text{-}3}(\nu-1)+2A_{1\text{-}2\text{-}3}(\nu-2)+3A_{1\text{-}2\text{-}3}(\nu-3),$$

we split the n-color 1-2-3 compositions of ν into three classes:

(A) Compositions having 1_1 on the right end;

(B) Compositions having 2_1 or 2_2 on the right end;

(C) Compositions having 3_1, 3_2 or 3_3 on the right end.

We transform the n-color 1-2-3 compositions in Class (A) by deleting 1_1 on the right end of compositions. This produces n-color 1-2-3 compositions enumerated by $A_{1\text{-}2\text{-}3}(\nu-1)$. Conversely, for each n-color 1-2-3 composition of $\nu-1$, we add 1_1 to the right end to produce an element in Class (A). In this way, we prove that there are exactly $A_{1\text{-}2\text{-}3}(\nu-1)$ compositions in Class (A).

Next, we transform the n-color 1-2-3 compositions in Class (B) by deleting 2_1 or 2_2 on the right end of compositions. This produces n-color 1-2-3 compositions of $\nu-2$. Because we can get n-color 1-2-3 compositions of $\nu-2$ by deleting 2_1 or 2_2, and vice versa, we see that the number of n-color 1-2-3 compositions in Class (B) is equal to $2A_{1\text{-}2\text{-}3}(\nu-1)$. Similarly, we can produce n-color 1-2-3 compositions enumerated by $3A_{1\text{-}2\text{-}3}(\nu-3)$ in Class (C) by deleting 3_1, 3_2 and 3_3 on the right end.

Hence, we have $A_{1\text{-}2\text{-}3}(\nu)=A_{1\text{-}2\text{-}3}(\nu-1)+2A_{1\text{-}2\text{-}3}(\nu-2)+3A_{1\text{-}2\text{-}3}(\nu-3)$.

Obviously, the relevant n-color compositions of 1 is (1_1), the relevant n-color compositions of 2 are $(1_1, 1_1)$, (2_1), (2_2), and the relevant n-color compositions of 3 are $(1_1, 1_1, 1_1)$, $(1_1, 2_1)$, $(1_1, 2_2)$, $(2_1, 1_1)$, $(2_2, 1_1)$, (3_1), (3_2), (3_3). This completes the proof.

2 *n*-color 1-2-3 palindromic compositions

Ref. [10] also studied the palindromic compositions with parts of size are less than 4, we refer to here as 1-2-3 palindromic compositions. Let $P_{1\text{-}2\text{-}3}(\nu)$ denote the number of 1-2-3 palindromic compositions of ν. Ref. [10] gave the generating function of $P_{1\text{-}2\text{-}3}(\nu)$ as $\dfrac{1+x+x^2+x^3}{1-x^2-x^4-x^6}$, which generates the interleaved generalized Tribonacci sequence 1, 1, 2, 2, 3, 3, 6, 6, 11, 11,….

From the generating function one can easily get the following recurrence relation.

Theorem 2.1 Let $P_{1\text{-}2\text{-}3}(\nu)$ denote the number of 1-2-3 palindromic compositions of ν. Then

$$P_{1\text{-}2\text{-}3}(1)=1,\ P_{1\text{-}2\text{-}3}(2)=2,\ P_{1\text{-}2\text{-}3}(3)=2,$$
$$P_{1\text{-}2\text{-}3}(4)=3,\ P_{1\text{-}2\text{-}3}(5)=3,\ P_{1\text{-}2\text{-}3}(6)=6,\ \text{and}$$
$$P_{1\text{-}2\text{-}3}(\nu)=P_{1\text{-}2\text{-}3}(\nu-2)+P_{1\text{-}2\text{-}3}(\nu-4)+P_{1\text{-}2\text{-}3}(\nu-6)\ \text{for}\ \nu>6.$$

We now show the combinatorial proof in this section.

Proof We split the 1-2-3 palindromic compositions of ν into three classes:

(A) Compositions having 1 on both extremes;

(B) Compositions having 2 on both extremes;

(C) Compositions having 3 on both extremes.

We transform the 1-2-3 palindromic compositions in Class (A) by deleting 1 on both extremes of compositions. This produces the 1-2-3 palindromic compositions enumerated by $P_{1\text{-}2\text{-}3}(\nu-2)$. Conversely, given any 1-2-3 palindromic composition enumerated by $P_{1\text{-}2\text{-}3}(\nu-2)$, we append 1 to both extremes of composition to produce the composition in Class (A). In this way, we establish the fact that there are exactly $P_{1\text{-}2\text{-}3}(\nu-2)$ elements in Class (A). Similarly, we can produce $P_{1\text{-}2\text{-}3}(\nu-4)$ elements in Class (B) by deleting 2 on both extremes of compositions, and obtain $P_{1\text{-}2\text{-}3}(\nu-6)$ elements in Class (C) by deleting 3 on both extremes of compositions.

Hence, we have $P_{1\text{-}2\text{-}3}(\nu)=P_{1\text{-}2\text{-}3}(\nu-2)+P_{1\text{-}2\text{-}3}(\nu-4)+P_{1\text{-}2\text{-}3}(\nu-6)$.

Clearly the initial conditions are true. So we complete the proof.

In this section, we further study the n-color 1-2-3 palindromic compositions.

Definition 2.1 An n-color 1-2-3 composition whose parts read from left to right are identical with when read from right to left is called an n-color 1-2-3 palindromic composition.

For example, there are 4 n-color 1-2-3 palindromic compositions of 3 as follows: 3_1, 3_2, 3_3, $1_1+1_1+1_1$.

We have the following result.

Theorem 2.2 Let $S_{1\text{-}2\text{-}3}(\nu)$ denote the number of n-color 1-2-3 palindromic compositions of ν. Then

$$S_{1\text{-}2\text{-}3}(1)=1,\ S_{1\text{-}2\text{-}3}(2)=3,\ S_{1\text{-}2\text{-}3}(3)=4,$$
$$S_{1\text{-}2\text{-}3}(4)=5,\ S_{1\text{-}2\text{-}3}(5)=6,\ P_{1\text{-}2\text{-}3}(6)=14,$$

and

$$S_{1\text{-}2\text{-}3}(\nu)=S_{1\text{-}2\text{-}3}(\nu-2)+2S_{1\text{-}2\text{-}3}(\nu-4)+3S_{1\text{-}2\text{-}3}(\nu-6)\ \text{for}\ \nu>6.$$

Here we provide a combinatorial proof.

Proof We split the n-color 1-2-3 palindromic compositions of v into three classes:

(A) Compositions having 1_1 on both extremes;

(B) Compositions having 2_1 or 2_2 on both extremes;

(C) Compositions having 3_1, 3_2 or 3_3 on both extremes.

We transform the n-color 1-2-3 palindromic compositions in Class (A) by deleting 1_1 on both extremes of palindromic compositions. This produces the n-color 1-2-3 palindromic compositions enumerated by $S_{1\text{-}2\text{-}3}(\nu-2)$. Conversely, given any n-color 1-2-3 palindromic composition of $\nu-2$, we append part of size 1_1 to both extremes of compositions to produce the compositions in Class (A). In this way, we establish the fact that there are exactly $S_{1\text{-}2\text{-}3}(\nu-2)$ elements in Class (A).

Next, we transform the n-color 1-2-3 palindromic compositions in Class (B) by deleting 2_1 or 2_2 on both extremes of compositions. This produces n-color 1-2-3 palindromic compositions of $\nu-4$. Since we can get n-color 1-2-3 palindromic compositions of $\nu-4$ by deleting 2_1 or 2_2 on both extremes, and vice versa, we see that the number of n-color 1-2-3 palindromic compositions in Class (B) is equal to $2S_{1\text{-}2\text{-}3}(\nu-4)$.

The operation of Class (C) follows as above, thus we can produce $3S_{1\text{-}2\text{-}3}(\nu-6)$ n-color 1-2-3 palindromic compositions in Class (C).

Hence, we have $S_{1\text{-}2\text{-}3}(\nu)=S_{1\text{-}2\text{-}3}(\nu-2)+2S_{1\text{-}2\text{-}3}(\nu-4)+3S_{1\text{-}2\text{-}3}(\nu-6)$, for $\nu>6$.

Obviously, the relevant n-color palindromic compositions of 1 is (1_1), the relevant n-color palindromic compositions of 2 are $(1_1, 1_1)$, (2_1), (2_2), the relevant n-color palindromic compositions of 3 are $(1_1, 1_1, 1_1)$, (3_1), (3_2), (3_3), the relevant n-color palindromic compositions of 4 are $(1_1, 2_1, 1_1)$, $(1_1, 2_2, 1_1)$, $(2_1, 2_1)$, $(2_2, 2_2)$, $(1_1, 1_1, 1_1, 1_1)$, the relevant n-color palindromic compositions of 5 are $(1_1, 3_1, 1_1)$, $(1_1, 3_2, 1_1)$, $(1_1, 3_3, 1_1)$, $(2_1, 1_1, 2_1)$, $(2_2, 1_1, 2_2)$, $(1_1, 1_1, 1_1, 1_1, 1_1)$, and the relevant n-color palindromic compositions of 6 are $(3_1, 3_1)$, $(3_2, 3_2)$, $(3_3, 3_3)$, $(2_1, 2_1, 2_1)$, $(2_2, 2_1, 2_2)$, $(2_1, 2_2, 2_1)$, $(2_2, 2_2, 2_2)$, $(1_1, 1_1, 2_1, 1_1, 1_1)$, $(1_1, 1_1, 2_2, 1_1, 1_1)$, $(1_1, 2_1, 2_1, 1_1)$, $(1_1, 2_2, 2_2, 1_1)$, $(2_1, 1_1, 1_1, 2_1)$, $(2_2, 1_1, 1_1, 2_2)$, $(1_1, 1_1, 1_1, 1_1, 1_1, 1_1)$.

This completes the proof.

From therecurrence relation of $S_{1\text{-}2\text{-}3}(\nu)$, we easily obtain the generating function of $S_{1\text{-}2\text{-}3}(\nu)$.

Corollary 2.1 Let $S_{1\text{-}2\text{-}3}(q)$ denote the generating function of $S_{1\text{-}2\text{-}3}(\nu)$. Then

$$S_{1\text{-}2\text{-}3}(q)=\frac{q+3q^2+3q^3+2q^4+3q^6}{1-q^2-2q^4-3q^6}.$$

3 A relation for the 1-2-3 compositions

Ref. [10] studied the compositions of a positive integer ν with parts omitting all multiples of size 3, and gave the generating function for the number of these compositions as

$$\frac{1-x^3}{1-x-x^2-x^3}=\sum_{n=0}^{\infty}T_{n+1}x^n-\sum_{n=0}^{\infty}T_{n-2}x^n,$$

where T_n is the n-th Tribonacci number. That is $C_{\neq 3k}(n)=T_{n+1}-T_{n-2}$. We denote the number of compositions with parts omitting all multiples of size 3 of a positive integer ν by $C_{\neq 3m}(\nu)$. Using the recurrence relation of T_n, the relation between T_n and $C_{1\text{-}2\text{-}3}(n)$, we obtain the following relation.

Theorem 3.1 Let $C_{1\text{-}2\text{-}3}(\nu)$ and $C_{\neq 3m}(\nu)$ denote the number of 1-2-3 compositions of ν and the number of compositions with parts omitting all multiples of size 3 of ν, respectively. Then

$$C_{\neq 3m}(\nu)=C_{1\text{-}2\text{-}3}(\nu-1)+C_{1\text{-}2\text{-}3}(\nu-2),$$

where $\nu>2$.

We now present the combinatorial bijection.

Proof Given any composition of ν: $\lambda=a_1+a_2+\cdots+a_l$, where $a_i\neq 3k$ for every i, $i=1,2,\cdots,l$. We consider the following two cases:

(a) when $a_l=3m+1, m\in Z^{>0}$;

(b) when $a_l=3m+2, m\in Z^{>0}$.

In Case (a), we first transform the composition λ by subtracting 1 from a_l, then replacing (a_l-1) by $3+3+\cdots+3$. Next we replace $a_t (1\leqslant t<l)$ with $3+3+\cdots+3+1$ when a_t is $3m+1, m\in Z^{>0}$ and replace $a_t (1\leqslant t<l)$ by $3+3+\cdots+3+2$ when a_t is $3m+2, m\in Z^{>0}$, respectively. For example, $4+2+1+7\to 4+2+1+6\to 3+1+2+1+3+3$. In this way, we produce the 1-2-3 composition with parts are less than 4 of $\nu-1$. Conversely, for each 1-2-3 composition of $\nu-1$, we first append part of size 1 to the right end, then adjoin all 3's with the part on the right of 3 to form a new part from left to right. Consequently, we get the composition of ν with parts omitting all multiples of 3. For example, $3+2+1+3+3+1+1\to 3+2+1+3+3+1+1+1\to 5+1+7+1+1$.

Hence, we proved that there are $C_{1\text{-}2\text{-}3}(\nu-1)$ compositions in Case (a).

In Case (b), we transform the composition λ by subtracting 2 from a_l and replacing (a_l-2) by $3+3+\cdots+3$. The rest of the proof follows as Case (a). The inverse operation is the same as Case (a) except to append 2 to the right end. So we yield $C_{1\text{-}2\text{-}3}(\nu-2)$ compositions in Case (b). For example, $1+5+7+4+2\to 1+5+7+4\to 1+3+2+3+3+1+3+1$; and $2+3+3+1+3+1\to 2+3+3+1+3+1+2\to 2+7+4+2$.

Thus we have

$$C_{\neq 3m}(\nu)=C_{1\text{-}2\text{-}3}(\nu-1)+C_{1\text{-}2\text{-}3}(\nu-2).$$

We complete the proof.

Using the same proving method, we get the generalized relation as follows.

Theorem 3.2 For the integer $k>1$. Let $C_{\leqslant k}(\nu)$ and $C_{\neq km}(\nu)$ denote the number of compositions of ν with parts are less than or equal to k and the number of compositions with parts omitting all multiples of size k of ν, respectively. Then

$$C_{\neq km}(\nu)=C_{\leqslant k}(\nu-1)+C_{\leqslant k}(\nu-2)+\cdots+C_{\leqslant k}(\nu-k+1),$$

where $\nu>k-1$, m is positive integer.

Consequently, the Theorem 3.1 is the special case of Theorem 3.2 when $k=3$. Besides, we easily get the following well-known relation when $k=2$ in Theorem 3.2.

Corollary 3.1 The number of compositions of ν having only odd parts equals the number of compositions of $\nu-1$ having only parts of size 1 or 2.

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near-perfect numbers of the form $2^{\alpha-1}p_1^2p_2^2$. The issue is an extension of some classical number theory problems, such as perfect numbers or odd perfect numbers, which are not easy. The main related studies can be seen in Refs.[3, 5, 8, 11, 12].

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