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A uniform asymptotic estimate for ruin probability of a discrete-time risk model with subexponential innovations

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Abstract: The recursive equation $T_n = X_n + T_{n-1}Y_n$ was considered, in which X_n and Y_n are two independent random variables, and T_{n-1} on the right-hand side is independent of (X_n, Y_n). Under the assumption that X_n follows a subexponential distribution with a nonzero lower Karamata index, and that (X_n, Y_n) fulfills a certain dependence structure, some asymptotic formulas were obtained for the tail probabilities of T_n in this equation.

Key words: asympotics; the lower Karamata index; subexponentiality; uniformly

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对于在次指数组下一种离散风险模型破产概率的一致渐近估计

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摘要:考虑递归等式 $T_n = X_n + T_{n-1}Y_n$,其中 X_n 和 Y_n 相互独立,等式右边的 T_{n-1} 独立于(X_n, Y_n). 假设 X_n 的分布函数属于次指数族,并且具有非零的下 Karamate 指数,同时(X_n, Y_n)满足一定的相依结构,对等式中 T_n 的尾部概率进行了估计.

关键词:渐近性;下 Karamata 指数;次指数族;一致性

0 Introduction

In this paper, for every $i \in \mathbb{N} = \{1, 2, \cdots\}$, we denote a real-valued random variable X_i the insurance company's net loss (the total amount of claims less premiums) within period i and a positive random variable Y_i the stochastic compound interest factor over the same time period. Then the stochastic values by time n of aggregate net losses of the insurance company are defined to be

$$T_n = X_n + T_{n-1}Y_n,\ n \in \mathbb{N} \tag{1}$$

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with T_0 being an arbitrarily random variable. In the existing literature, $\{X_i, i\in\mathbb{N}\}$ and $\{Y_i, i\in\mathbb{N}\}$ are actually called the insurance risk and the financial risk, respectively. At the same time, T had been studied in insurance mathematics under the name perpetuity. Since schemes like relation (1) are ubiquitous in many areas of applied mathematics, the properties of T have attracted a considerable interest. From the application point of view, the key information is the behavior of the tail of T_n, that is $P(T>x)$, as $x\to\infty$. This problem was investigated by various authors, for example, by Goldie et al.[13], and in a similar setting by Hitczenko and Wesolowski[14]. The first result indicates that if X is bounded, $P(Y\in[0,1])=1$ and the distribution of Y behaves like the uniform distribution in the neighborhood of 1, then T has thin tail.

Ruin probabilities of such a discrete-time risk model have been extensively studied by many authors. However, most of the researches assumed that the sequences $\{X_i, i\in\mathbb{N}\}$ and $\{Y_i, i\in\mathbb{N}\}$ are mutually independent. Such an independence assumption was proposed mainly for the mathematical tractability rather than the practical relevance. Therefore, in recent years, more and more researchers have started to improve the model through introducing suitable dependence structures between the insurance risk and the financial risk. Yang and Wang[15] studied a model with (X, Y) following a bivariate Sarmonov distribution, and Qu and Chen[16] considered another type of dependence structure assuming that $(X_1,\cdots,X_n,Y_1,\cdots,Y_n)$ follows a multivariate Samanow distribution.

Motivated by Chen[17] and Yang and Wang[15], in this paper, we shall assume that (X_n,Y_n), $n\in\mathbb{N}$ is a positive random pair with the generic vector (X,Y) following a certain dependence structure, and, X follows a subexponential distribution with a nonzero lower Karamata index. Besides, we also assume that $EY^\beta<1$ with $0<\beta<J_F^*$. In order to make Y not necessarily bounded, we should consider the product

$$Z=XY \tag{2}$$

in which X and Y are independent random variables and distributed by F and G respectively. Denote H the distribution of Z. The subexponentiality of (2) was first studied by Cline and Samorodnitsky[10], and then was extended by Tang[4].

The rest of this paper is organized as follows. In Section 1 we will briefly recall basic notations and properties of subexponential distributions, and introduce the dependence structure that we will use in this paper. In Section 2, we will present a precise statement of our main results. Finally, we will give the full proof of the three results in Section 3, Section 4 and Section 5, respectively.

1 Preliminaries

1.1 Notational conventions

Throughout this paper, all limiting relationships hold for $x\to\infty$ unless otherwise stated. For two positive functions $f(x)$ and $g(x)$ satisfying

$$a\leqslant \liminf\frac{f(x)}{g(x)}\leqslant \limsup\frac{f(x)}{g(x)}\leqslant b,$$

we write $f(x)=O(g(x))$ if $b<\infty$, write $f(x)=o(g(x))$ if $a=b=0$, write $f(x)\lesssim g(x)$ if $b=1$, write $f(x)\gtrsim g(x)$ if $a=1$, and write $f(x)\sim g(x)$ if $a=b=1$.

1.2 Subexponential distributions

Due to important applications to real-world problems, we restrict our interest to the case of heavy-tailed distributions, as has recently been done by many researchers in applied probability and related fields. Specifically, we shall assume that the innovation X_n in Eq. (1) follows a subexponential distribution. By definition, a distribution function F on $(0,\infty)$ is said to be subexponential (hence, heavy-tailed), denoted by $F\in S$, if the right tail of F is infinite (that is $\overline{F}(x)>0$), and the relation

$$\overline{F^{2*}(x)}\sim 2\overline{F}(x) \tag{3}$$

holds, where $F^{2*}(x)$ denotes the 2-fold convolution of F. The authoritative narration about subexponential distributions can be found in the monographs by Embrechts et al.[1] and Foss et al.[2].

It is well-known that, if $F \in S$, then it is long-tailed, denoted by $F \in \mathscr{L}$, which means the relation

$$\lim_{x\to\infty}\frac{\overline{F}(x-t)}{\overline{F}(x)}=1 \tag{4}$$

holds uniformly on compact t-sets of $(0,\infty)$; see Ref.[1, Lemma 1.3.5] for more details.

For two distribution functions F_1 and F_2 on $(0,\infty)$, denote $F=F_1 * F_2$, that is F is the convolution of F_1 and F_2. If $F_1 \in \mathscr{S}$, $F_2 \in \mathscr{L}$, and $\overline{F_2}(x)=O\,\overline{F_1}(x)$, then $F\in\mathscr{S}$ and

$$\overline{F}(x) \sim \overline{F_1}(x)+\overline{F_2}(x) \tag{5}$$

For details, see Refs.[5,6,11].

In this paper, a distribution F on $(-\infty,+\infty)$ is still said to be subexponential if the distribution $F_+(x)=F(x)I(x\geqslant 0)$ is subexponential. In this more general sense, the subexponentiality still implies relation (3), for details, see Ref.[18, Lemma 2.2].

1.3 The Karamata index

To establish exact asymptotic formulas for the tails of T_n in (1), we need to impose a technical assumption on the tail of X_n. For a positive function $f(x)$, its lower Karamata index J_F^* is the supremum of those α for which, for every $\Lambda>1$,

$$\frac{f(\lambda x)}{f(x)}\geqslant[1+o(1)]\lambda^{\alpha} \tag{6}$$

holds uniformly in $\lambda\in[1,\Lambda]$, for $x\to\infty$. For a distribution function F with an infinite right tail, that is $\overline{F}(x)>0$, write $f=1/\overline{F}$, we simply call J_F^* the lower Karamata index of F. For details, see Ref.[3, Subsection 2.1].

According to Ref.[3, Subsection 2.1], there is a closely related index Matuszewska index, denoted by M_F^*, the definition of which is as follows: For a positive function $f(x)$ ($f=1/\overline{F}$ for $\overline{F}>0$ as stated above), the M_F^* is the supremum of those β, for which for some $D=D(\beta)>0$ and all $\Lambda>1$, the relation

$$\frac{f(\lambda x)}{f(x)}\geqslant D\{1+o(1)\}\lambda^{\beta} \tag{7}$$

holds uniformly in $\lambda\in[1,\Lambda]$, for $x\to\infty$.

There are some properties for these two indexes. First, for any distribution function F with an infinite tail, we have $0\leqslant J_F^*\leqslant\infty$ due to the monotonicity of f. Second, if two distribution functions F_1 and F_2 have equivalent tail, that is, $\overline{F_1}(x)\sim c\,\overline{F_2}(x)$ for some $c>0$, then their lower Karamata indexs are equal. Third, according to the definitions, it is clear that $J_F^*\leqslant M_F^*$. For the class of all subexponential distributions with a nonzero lower Matuszewska index, which Tang[4] named the class $\mathscr{A}$. Similarly, for the class of all subexponential distributions with a nonzero lower Karamata index, we can use the class $\mathscr{A}^*$ to name it, then class $\mathscr{A}^*$ is marginally smaller than the class $\mathscr{A}$.

In this paper, if we want to prove a distribution function belongs to class $\mathscr{S}$, we just have to prove the relation $\overline{F*F}(x)\sim 2\overline{F}(x)$ holds. If we want to prove the distribution also belongs to class $\mathscr{A}^*$, we just have to prove the relation

$$\liminf_{x\to\infty}\frac{\overline{F}(x)}{\overline{F}(\lambda x)}>1 \text{ for } \lambda>1 \tag{8}$$

holds, which is the result of the definition of the lower Karamata index and class $\mathscr{A}^*$.

1.4 A dependence structure

Definition 1.1 A random pair (X,Y) with X unbounded is said to satisfy the dependence structure $\mathscr{H}$ if there is some positive and eventually bounded measurable function $h(x)$: $(0,\infty)\to(0,\infty)$, such that the asymptotic relation

$$P(X>x\mid Y=y)\sim h(y)P(X>x) \tag{9}$$

holds uniformly for all $y\in R(Y)$, where $R(Y)$ is the range of Y, consisting of all possible values of Y.

When y is not a possible value of Y, that is $y\notin R(Y)$, the conditional probability can simply be understood as unconditional, therefore $h(y)=1$. This dependence structure was first proposed by Asimit and Badescu[7], and was further studied and extended by Li et al. [8] and Yuen[9].

2 Main results

According to Ref. [4, Theorem 2.1 and Corollary 2.1], we have

Theorem 2.1 Consider the relation $Z=XY$ with F on $(-\infty,+\infty)$ and G on $(0,+\infty)$. We have that $H\in\mathscr{A}^*$ if $F\in\mathscr{A}^*$ and $\overline{G}(vx)=o(\overline{H}(x))$ for each $v>0$, where X and Y fulfill the dependence structure $\mathscr{H}$.

The result of Theorem 2.1 is the foundation of the following two results, and the core of the following result is the tail behavior of T_n.

Theorem 2.2 Consider the recursive equation (1) starting with $T_0=0$, in which the innovation pairs $(X_1,Y_1),\cdots,(X_n,Y_n)$ are independent, but not necessarily identically distributed. Assume that, for each $i=1,2,\cdots,n$, there is

① $P(X_i>x)\sim c_i\overline{F}(x)$ for some distribution function $F\in\mathscr{A}^*$ with a Karamata index $0<J_F^*\leqslant\infty$, and some $c_i>0$;

② $Y_i>0$ and is not necessarily bounded, and there is $\overline{G_i}(vx)=o(\overline{H_i}(x))$ and $\overline{G_i}(vx)=o(\overline{F_i}(x))$ for each $v>o$;

③ (X_i,Y_i) fulfills the dependence structure $\mathscr{H}$ with auxiliary function h_i.

Then

$$P(T_n>x)\sim\sum_{i=1}^{n}P(X_i\prod_{j=i+1}^{n}Y_j>x)\quad(10)$$

The proof of Theorem 2.2 will be built on the recursive equation (1). The following result, interesting in its own right, describes the tail behavior of T_n given by (1) and will be used to establish the finite-term version of Theorem 2.2.

Theorem 2.3 Consider the recursive equation (1) starting with $T_0=0$, in which the innovation pairs $(X_n,Y_n),n\in\mathbb{N}$ are i.i.d. copies of a positive pair (X,Y). Assume the following:

① X is distributed by $F\in\mathscr{A}^*$ with a lower Karamata index $0<J_F^*\leqslant\infty$;

② $EY^\beta<1$ for $0<\beta<J_F^*$ and its distribution function satisfies $\overline{G}(vx)=o(\overline{H}(x))$ and $\overline{G}(vx)=o(\overline{F}(x))$ for each $v>0$;

③ (X,Y) fulfills the dependence structure $\mathscr{H}$.

Then the relation

$$P(T_n>x)\sim\sum_{i=1}^{n}P(X_i\prod_{j=1}^{i-1}Y_j>x)$$

holds uniformly for $n\in\mathbb{N}$. Particularly, when $n=\infty$, the relation

$$P(T_\infty>x)\sim\sum_{i=1}^{\infty}P(X_i\prod_{j=1}^{i-1}Y_j>x)\quad(11)$$

holds.

3 Proof of Theorem 2.1

3.1 Lemmas

Similarly with Ref.[4, Lemmas 3.1, 3.2], we can prove the following lemma:

Lemma 3.1 Let X_1 and X_2 be i.i.d random variables with common distribution $F\in\mathscr{A}^*$, then we have

$$\lim_{x\to\infty}\sup_{0\leqslant r\leqslant1}\frac{P(X_1+rX_2>x)}{P(X_1>x)+P(rX_2>x)}=1\quad(12)$$

The following lemma will play crucial roles in the proof of Theorem 2.2.

Lemma 3.2 Consider two distribution functions F and G mentioned above, and assume X and Y satisfy the dependence structure $\mathscr{H}$. If we have $\overline{G}(vx)=o(\overline{H}(x))$ for each $v>0$, then we can have the relation

$$P(XY>x)\sim\int_0^\infty h(y)P(X>\frac{x}{y})P(Y\in\mathrm{d}y)\quad(13)$$

Proof Since Y is nonnegative, we have that

$$P(XY>x)=P(X^+Y>x)\quad(14)$$

holds for all $x>0$, where $X^+=XI(X>0)$.

According to Ref. [4, Lemma 3.2], there is a function $a(x):[0,\infty)\to[0,\infty)$ satisfying:

① $a(x)\nearrow\infty$;

② $a(x)/x\searrow0$;

③ $\overline{G}(a(x))=o(\overline{H}(x))$.

Hence, we have

$$P(XY>x)=\left(\int_0^{a(x)}+\int_{a(x)}^\infty\right)P(X>\frac{x}{y}\mid Y=y)G(\mathrm{d}y)=\int_0^{a(x)}P(X>\frac{x}{y}\mid Y=y)G(\mathrm{d}y)+O(\overline{G}(a(x)))\sim$$

$$\int_0^{a(x)} h(y)P(X > \frac{x}{y})G(\mathrm{d}y) + o(P(XY > x)) =$$
$$\left(\int_0^{\infty} - \int_{a(x)}^{\infty}\right) h(y)P(X > \frac{x}{y})G(\mathrm{d}y) + o(\overline{H}(x)) =$$
$$\int_0^{\infty} h(y)P(X > \frac{x}{y})G(\mathrm{d}y).$$

The last step is driven by that $h(y)$ is eventually bounded for $y \in R(Y)$, so we have

$$\int_{a(x)}^{\infty} h(y)\overline{F}(\frac{x}{y})G(\mathrm{d}y) = o(\overline{H}(x)).$$

This ends the proof.

We can find a random variable Y^* which is independent of X and distributed by

$$P(Y^* \in \mathrm{d}y) = h(y)P(Y \in \mathrm{d}y)$$

where $Y \in (0, \infty]$. By the uniformity required in the dependence structure $\mathscr{H}$, integrating both sides of Eq.(9) with respect to $G(\mathrm{d}y)$ over range of $(0, \infty)$ leads to

$$E(h(y)) = \int_0^{\infty} h(y)G(\mathrm{d}y) = 1,$$

then we have

$$P(XY > x) \sim P(XY^* > x) \tag{15}$$

At the same time, we can also have $\overline{G^*}(vx) = o(\overline{H^*}(x))$ for all $v > 0$, where G^* is the distribution function of Y^*, and H^* is the distribution function of the product of XY.

3.2 Proof of Theorem 2.1

Hereafter, we write $x^+ = x_+ = \max\{x, 0\}$ for every real number x. We prove that $H \in \mathscr{S}$. By definition, it suffices to prove the relation

$$\overline{H_+ * H_+}(x) \lesssim 2\,\overline{H_+}(x) \tag{16}$$

Since the reverse relation is automatic for every distribution on $[0, \infty)$. By Ref. [4, Lemma 3.2], there exists a function $a(x)$ satisfying the three conditions mentioned above. Let X_i and Y_i, $i = 1, 2$, be independent copies of X and Y respectively. We have

$$\overline{H_+ * H_+}(x) = P(X_1^+Y_1 + X_2^+Y_2 > x,$$
$$(Y_1 > a(x)) \cup (Y_2 > a(x))) +$$
$$P(X_1^+Y_1 + X_2^+Y_2 > x, 0 < Y_2 \leqslant Y_1 \leqslant a(x)) +$$
$$P(X_1^+Y_1 + X_2^+Y_2 > x, 0 < Y_1 < Y_2 \leqslant a(x)) =$$
$$J_1(x) + J_2(x) + J_3(x) \tag{17}$$

It is clear that $J_1(x) \leqslant 2\overline{G}(a(x))$, so we have $J_1(x) = o(\overline{H}(x))$.

By Lemma 3.1, we have

$$J_2 = \iint_{0 \leqslant y_2 \leqslant y_1 \leqslant a(x)} P(X_1^+ + \frac{y_2}{y_1}X_2^+ > \frac{x}{y_1} \mid Y_1 = y_1,$$
$$Y_2 = y_2)G(\mathrm{d}y_1)G(\mathrm{d}y_2) \leqslant$$
$$\iint_{0 \leqslant y_2 \leqslant y_1 \leqslant a(x)} (P(X_1 > \frac{x}{y_1} \mid Y_1 = y_1) +$$
$$P(X_2 > \frac{x}{y_2} \mid Y_2 = y_2))G(\mathrm{d}y_1)G(\mathrm{d}y_2) \sim$$
$$\iint_{0 \leqslant y_2 \leqslant y_1 \leqslant a(x)} (h(y_1)P(X_1 > \frac{x}{y_1}) +$$
$$h(y_2)P(X_2 > \frac{x}{y_2}))G(\mathrm{d}y_1)G(\mathrm{d}y_2) =$$
$$P(X_1Y_1^* > x, 0 < Y_2 \leqslant Y_1^* \leqslant a(x)) +$$
$$P(X_2Y_2^* > x, 0 < Y_2^* \leqslant Y_1 \leqslant a(x)).$$

Similarly we have

$$J_3(x) \lesssim P(X_1Y_1^* > x, 0 < Y_1^* < Y_2 \leqslant a(x)) +$$
$$P(X_2Y_2^* > x, 0 < Y_1 < Y_2^* \leqslant a(x)).$$

Then we have

$$\overline{H_+ * H_+}(x) = J_1(x) + J_2(x) + J_3(x) \lesssim$$
$$o(\overline{H}(x)) + 2\,\overline{H^*}(x) \sim 2\overline{H}_+(x).$$

This proves relation (16).

Then similarly with relation (14), we have $\overline{H * H}(x) \sim 2\overline{H}(x)$. Next we prove $H \in \mathscr{A}^*$. There is some constant $\lambda > 1$ such that relation (8) holds with F belonging to class $\mathscr{A}^*$. Then, with this constant Λ and the function $a(x)$ defined in Ref. [4, Lemma 3.2], we have

$$\liminf_{x\to\infty} \frac{\overline{H}(x)}{\overline{H}(\lambda x)} = \liminf_{x\to\infty} \frac{\left(\int_0^{a(x)} + \int_{a(x)}^{\infty}\right) P(X > x/y \mid Y = y)G(\mathrm{d}y)}{\overline{H}(\lambda x)} \geqslant$$
$$\liminf_{x\to\infty} \frac{\int_0^{a(x)} \overline{F}(x/y)G^*(\mathrm{d}y)}{\int_0^{a(x)} \overline{F}(\lambda x/y)G^*(\mathrm{d}y) + \int_{a(x)}^{\infty} P(X > \lambda x/y \mid Y = y)G(\mathrm{d}y)} \sim$$

$$\liminf_{x\to\infty}\frac{\int_0^{a(x)}\overline{F}(x/y)G^*(\mathrm{d}y)}{\int_0^{a(x)}\overline{F}(\lambda x/y)G^*(\mathrm{d}y)+o(\overline{H}^*(\lambda x))}=\liminf_{x\to\infty}\frac{\int_0^{a(x)}\overline{F}(x/y)G^*(\mathrm{d}y)}{(1+o(1))\int_0^{a(x)}\overline{F}(\lambda x/y)G^*(\mathrm{d}y)}\geqslant$$

$$\liminf_{x\to\infty}\inf_{0<y\leqslant a(x)}\frac{\overline{F}(x/y)}{\overline{F}(\lambda x/y)}>1.$$

So we have $\mathscr{H}\in\mathscr{A}^*$. This ends the proof of Theorem 2.1.

4 Proof of Theorem 2.2

We use the mathematical induction device.

Proof For $n=1$, $T_1=X_1$, the relation (10) holds trivially and the distribution function of T_1 belongs to class $\mathscr{A}^*$.

For $n=2$, $T_2=X_2+T_1Y_1=X_2+X_1Y_2$.

According to Theorem 2.1, we have the distribution function of X_2Y_2 belonging to class $\mathscr{A}^*$ under the conditions above, and we can find a function $a(x)$, so that

$$P(X_1Y_2>x)=\left(\int_0^{a(x)}+\int_{a(x)}^{\infty}\right)\overline{F_1}\left(\frac{x}{y_2}\right)G_2(\mathrm{d}y_2)=$$

$$\int_0^{a(x)}\frac{c_1}{c_2}\overline{F_2}\left(\frac{x}{y_2}\right)G_2(\mathrm{d}y_2)+O(\overline{G_2}(a(x)))\sim$$

$$\frac{c_1}{c_2h_2(y_2)}P(X_2Y_2>x,\ 0\leqslant Y_2\leqslant a(x)+o(\overline{H_2}(x))=$$

$$(O(1)+o(1))\overline{H_2}(x).$$

According to Lemma A3.15 (closure of $\mathscr{S}$ under tail equivalence) of Embrechts et al.[1], the distribution function of X_1Y_2 belongs to class $\mathscr{S}$, that is, the distribution function of T_1Y_2 belongs to class $\mathscr{S}$. At the same time, we can also find that if $Y_2\leqslant 1$, then $P(X_1Y_2>x)=O(1)P(X_2>x)$; and if $Y_2>1$, then $P(X_2>x)=O(1)P(X_1Y_2>x)$. According to relation (5), we have the distribution of T_2 belonging to class $\mathscr{S}$ and the relation (10) holds.

Next, we prove the distribution function of T_2 belongs to class $\mathscr{A}^*$:

$$\liminf_{x\to\infty}\frac{P(T_2>x)}{P(T_2>\lambda x)}\sim$$

$$\liminf_{x\to\infty}\frac{P(X_2>x)+P(X_1Y_2>x)}{P(X_2>\lambda x)+P(X_1Y_2>\lambda x)}>1,$$

the last step is driven by

$$\liminf_{x\to\infty}\frac{P(X_1Y_2>x)}{P(X_1Y_2>\lambda x)}\sim\liminf_{x\to\infty}\frac{P(X_2Y_2>x)}{P(X_2Y_2>\lambda x)}>1.$$

We proceed by induction on n: Note that

(1:1) $T_1Y_2\in\mathscr{S}$;

(1:2) $T_2\in\mathscr{A}^*$;

(1:3) the relation (10) holds for $n=2$.

Now we assume that:

(m:1) $T_{m-1}Y_m\in\mathscr{S}$;

(m:2) $T_m\in\mathscr{A}^*$;

(m:3) the relation (10) holds for $n=m$.

We aim to prove that:

($m+1$:1) $T_mY_{m+1}\in\mathscr{S}$;

($m+1$:2) $T_{m+1}\in\mathscr{A}^*$;

($m+1$:3) the relation (10) holds for $n=m+1$.

First, we prove ($n+1$:1):

For each $v>0$,

$$\limsup_{x\to\infty}\frac{P(Y_{m+1}>vx)}{P(T_m>x)}=$$

$$\limsup_{x\to\infty}\frac{P(Y_{m+1}>vx)}{P(X_m+T_{m-1}Y_m>x)}\sim$$

$$\limsup_{x\to\infty}\frac{P(Y_{m+1}>vx)}{P(X_m>x)+P(T_{m-1}Y_m>x)}\leqslant$$

$$\limsup_{x\to\infty}\frac{P(Y_{m+1}>vx)}{P(X_m>x)}\to 0 \tag{18}$$

According to Theorem 2.1, and let $h(x)\equiv 1$, that is T_m and Y_{m+1} are independent, we have the distribution function of T_mY_{m+1} belonging to class $\mathscr{A}^*$.

Next, we prove ($m+1$:2):

When $Y_{m+1}<1$, we can prove that

$$P(T_{m-1}Y_mY_{m+1}>x)=O(1)P(X_mY_{m+1}>x),$$

then we have

$$\limsup_{x\to\infty}\frac{P(T_mY_{m+1}>x)}{P(X_{m+1}>x)}=$$

$$\limsup_{x\to\infty}\left[\frac{P(X_mY_{m+1}>x)}{P(X_{m+1}>x)}+\frac{P(T_{m-1}Y_mY_{m+1}>x)}{P(X_{m+1}>x)}\right]<\infty,$$

that is,

$$P(T_m Y_{m+1} > x) = O(1) P(X_{m+1} > x).$$

When $Y_{m+1} \geqslant 1$

$$\limsup_{x\to\infty} \frac{P(X_{m+1} > x)}{P(T_m Y_{m+1} > x)} = \limsup_{x\to\infty} \frac{P(X_{m+1} > x)}{P(X_m Y_{m+1} + T_{m-1} Y_m Y_{m+1} > x)} \leqslant \frac{P(X_{m+1} > x)}{P(X_m Y_{m+1} > x)} < \infty,$$

then we have

$$P(X_{m+1} > x) = O(1) P(T_m Y_{m+1} > x).$$

Thus the distribution function of T_{m+1} belongs to class $\mathscr{S}$, and it is easy to check that the distribution function of the sum $X_{m+1} + T_m Y_{m+1}$, which is the distribution function of T_{m+1} belongs to class $\mathscr{A}^*$.

Finally, we prove $(m+1;3)$:

$$P(T_{m+1} > x) \sim P(X_{m+1} > x) + \left(\int_0^{a(x)} + \int_{a(x)}^{\infty}\right) P\left(T_m > \frac{x}{y_{m+1}}\right) G_{m+1}(\mathrm{d}y_{m+1}) = P(X_{m+1} > x) + (1+o(1)) \int_0^{a(x)} \sum_{i=1}^{m} P\left(x_i \prod_{j=i+1}^{m} Y_j > \frac{x}{y_{m+1}}\right) G_{m+1}(\mathrm{d}y_{m+1}) = (1+o(1)) \sum_{i=1}^{m+1} P\left(X_i \prod_{j=i+1}^{m+1} Y_j > x\right).$$

The second step is due to the relation (18). Hence, relation (10) holds for $n = m+1$.

This ends the proof of Theorem 2.2.

5 Proof of Theorem 2.3

5.1 Lemmas

The lemma below describes an important property of distributions with a nonzero lower Karamata index.

Lemma 5.1 Let F be a distribution function with a lower Karamata index $0 < J_F^* \leqslant \infty$, and the second statement of Ref.[3, Proposition 2.23] tells us that for each $0 < \beta < J_F^*$, and $A \in (0,1)$, there exists $x_0 = x_0(A, \beta)$ such that the inequality

$$\frac{f(x/y)}{f(x)} \geqslant A y^{-\beta}$$

holds uniformly for $x/y \geqslant x \geqslant x_0$, or equivalently, that the inequality

$$\frac{\overline{F}(x/y)}{\overline{F}(x)} \leqslant a y^{\beta} \tag{19}$$

holds with $a = 1/A$ uniformly for $x/y \geqslant x \geqslant x_0$.

Plugging in inequality (19) $x = x_0$, $y = x_0/t$ for large t, we see that, for some constant $c > 0$,

$$\overline{F}(t) \leqslant c t^{-\beta}.$$

Another immediate consequence of Lemma 5.1 is as follows:

Lemma 5.2 Let X be a random variable distributed by F with a lower Karamata index $0 < J_F^* \leqslant \infty$. Then for every $0 < \beta < J_F^*$ and every $a > 1$, there is some $x_0 = x_0(a, \beta) > 0$, such that, for all $x > x_0$ and Y is independent of X,

$$\frac{P(XY > x)}{P(X > x)} \leqslant a E Y^{\beta}.$$

Proof

$$\frac{P(XY > x)}{P(X > x)} = \frac{\left(\int_0^{a(x)} + \int_{a(x)}^{\infty}\right) P\left(X > \frac{x}{y}\right) P(Y \in \mathrm{d}y)}{P(X > x)} \leqslant \int_0^{a(x)} a y^{\beta} P(Y \in \mathrm{d}y) \leqslant a E Y^{\beta}.$$

For two random variables X and Y, we use the notation $X \leqslant_d Y$ to denote that X is stochastically dominated by Y, that is, $P(X > x) \leqslant P(Y > x)$ for all real x. Motivated by Ref.[12], we establish the following lemma, which will play a crucial role in the proof of Theorem 2.3.

Lemma 5.3 Let (X, Y) satisfy the dependence structure $\mathscr{H}$, and let $F \in \mathscr{A}^*$ with a lower Karamata index $0 < J_F^* \leqslant \infty$. Let $EY^{\beta} < 1$ for $0 < \beta < J_F^*$ and its distribution function satisfy $\overline{G}(vx) = o(\overline{H}(x))$ for each $v > 0$. Then there is some positive random variable Z independent of (X, Y) with $P(Z > x) \sim \alpha \overline{F}(x)$ for some large $\alpha > 0$, such that

$$X + ZY \leqslant_d Z \tag{20}$$

Proof For some $b > 0$ to be specified later, we choose some random variable Z_b independent of (X, Y) such that

$$P(Z_b > x) \sim b \overline{F}(x) \tag{21}$$

Then the distribution function of Z_b also belongs to class $\mathscr{A}^*$ and has the same lower Karamata index J_F^*, then

$$P(X + Z_b Y > x) \sim$$

$$P(X>x)+P(Z_bY>x)\leqslant$$
$$(1+abEY^\beta)\overline{F}(x)$$

where we choose a small enough and $b>0$ large enough such that $aEY^\beta<1$ and $1+abEY^\beta<b$. The first step is due to Ref.[4] and the relation (21), so we can conclude that there is some $x_0>0$ such that the inequality

$$P(X+Z_bY>x)\lesssim P(Z_b>x) \tag{22}$$

holds for $x>x_0$.

Finally, we construct the random variable Z as follows:

$$P(Z>x)=P(Z_b>x\mid Z_b>x_0)=$$
$$\begin{cases}1, & x\leqslant x_0;\\ P(Z_b>x)/P(Z_b>x_0), & x>x_0.\end{cases}$$

By inequality (22), it is easy to check that inequality (20) holds. This ends the proof.

5.2 Proof of Theorem 2.3

Keep in mind that the limit T_∞ is irrespective to T_0. Let Z be a positive random variable specified in Lemma 5.3 and independent of $\{(X_i, Y_i), i=1,2,\cdots\}$. If T_0 is indentified as Z, then applying Lemma 5.3 to the recursive equation (1) we have

$$T_n\leqslant_d Z \text{ for every } n\in\mathbb{N}\cup\{\infty\} \tag{23}$$

Now let the recursive equation (1) start with $T_0=0\leqslant_d Z$, so that (23) is still valid. For every $n\in\mathbb{N}$, since $\{(X_1,Y_1),\cdots(X_n,Y_n)\}$ and $\{(X_n, Y_n), \cdots(X_1, Y_1)\}$ are equal in distribution, we have

$$T_n=\sum_{i=1}^{n}X_i\prod_{j=i+1}^{n}Y_j=\sum_{i=1}^{n}X_i\prod_{j=1}^{i-1}Y_j=\widetilde{T}_n \tag{24}$$

For arbitrarily fixed $N\in\mathbb{N}$, by Theorem 2.2 we have

$$P(T_n>x)\sim\sum_{i=1}^{N}P(X_i\prod_{j=i+1}^{N}Y_j)=$$
$$\sum_{i=1}^{N}P(X_i\prod_{j=1}^{i-1}Y_j>x).$$

For every $n>N$, first we aim at a uniform upper bound for the tail probability of T_n. For this purpose, split $\widetilde{T}_n$ in (24) into two parts as

$$\widetilde{T}_n=\sum_{i=1}^{N}X_i\prod_{j=1}^{i-1}Y_j+(\sum_{i=N+1}^{n}X_i\prod_{j=N+1}^{i-1}Y_j)\prod_{j=1}^{N}Y_j \tag{25}$$

Note that

$$\sum_{i=N+1}^{n}X_i\prod_{j=N+1}^{i-1}Y_j=_d\widetilde{T}_{n-N}=_d T_{n-N}\leqslant_d Z.$$

Thus,

$$\widetilde{T}_n\leqslant_d\sum_{i=1}^{N}X_i\prod_{j=1}^{i-1}Y_j+Z\prod_{j=1}^{N}Y_j \tag{26}$$

It follows (26) that

$$P(T_n>x)=P(\widetilde{T}_n>x)\leqslant$$
$$P(\sum_{i=1}^{N}X_i\prod_{j=1}^{i-1}Y_j+Z\prod_{j=1}^{N}Y_j>x)\sim$$
$$\sum_{i=1}^{n}P(X_i\prod_{j=1}^{i-1}Y_j>x)+P(Z\prod_{j=1}^{N}Y_j>x)\leqslant$$
$$\sum_{i=1}^{n}P(X_i\prod_{j=1}^{i-1}Y_j>x)+P(Z\prod_{j=1}^{N}Y_j>x) \tag{27}$$

By Lemma 5.2, for every $0<\beta<J_F^*$ and every $a>1$, there is some $x_0=x_0(a,\beta)>0$ irrespective to N such that, for all $x>x_0$,

$$P(Z\prod_{j=1}^{N}Y_j>x)\leqslant aE[\prod_{j=1}^{N}Y_j]^\beta P(Z>x)\sim$$
$$(\alpha a\mid[EY^\beta]^N)\overline{F}(x)$$

where the last step holds for some large constant $K>0$. For arbitrarily given small $\delta>0$, since $EY^\beta<1$, we can choose some $N=N_1$ sufficiently large such that the prefactor of $\overline{F}(x)$ is not greater than δ. It follows (27) that, for all $n>N_1$,

$$P(T_n>x)\lesssim(1+\delta)\sum_{i=1}^{n}P(X_i\prod_{j=1}^{i-1}Y_j>x) \tag{28}$$

Next we derive a uniform lower bound for the tail probability of T_n. Still starting form Eqs.(24) and (25), we have for all $n>N$,

$$T_n=_d\widetilde{T}_n\leqslant\sum_{i=1}^{N}X_i\prod_{j=1}^{i-1}Y_j=_d T_n$$

Applying Theorem 2.2, it follows that

$$P(T_n>x)\geqslant P(T_N>x)\sim\sum_{i=1}^{N}P(X_i\prod_{j=i+1}^{N}Y_j>x)=$$
$$(\sum_{i=1}^{n}-\sum_{i=N+1}^{n})P(X_i\prod_{j=1}^{i-1}Y_j>x) \tag{29}$$

Similar to the above, by Lemma 5.2, for every $0<\beta<J_F^*$ and every $a>1$, there is some $x_0=x_0(a,\beta)>0$, such that for all $x>x_0$ and all $n>N$,

$$\sum_{i=N+1}^{n}P(X_i\prod_{j=1}^{i-1}Y_j>x)\leqslant a\overline{F}(x)\sum_{i=N+1}^{n}[EY^\beta]^{i-1}=$$

$$a\,\frac{[EY^{\beta}]^{N}}{1-EY^{\beta}}\overline{F}(x)$$

Thus, for arbitrarily given small $\delta>0$, we can find some $N=N_2$ sufficiently large such that the prefactor of $P(XY>x)$ above is not greater than δ. It follows (29) that, for all $N>N_2$,

$$P(T_n>x)\gtrsim(1-\delta)\sum_{i=1}^{n}P(X_i\prod_{j=1}^{i-1}Y_j>x) \tag{30}$$

Since, by Theorem 2.2, both relation (28) and (30) hold for $1\leqslant n\leqslant N_1\vee N_2$, so we complete the proof.

References

[1] EMBRECHTS P, KLUPPELBERG C, MIKOSCH T. Modelling Extremal Events for Insurance and Finance [M]. Berlin: Springer-Verlag, 1997.

[2] FOSS S, KORSHUNOV D, ZACHARY S. An Introduction to Heavy-Tailed and Subexponential Distribution[M]. 2nd ed. New York: Springer, 2013.

[3] BINGHAM N H, GOLDIE C M, TEUGELS J L. Regular Variation [M]. Cambridge: Cambridge University Press, 1987.

[4] TANG Q. The subexponentiality of products revisited [J]. Extremes, 2006, 9: 231-241.

[5] Embrechts P, Goldie C M. On closure and factorization properties of subexponential and related distributions [J]. J Aust Math Soc (Ser A), 1980, 29(2): 243-256.

[6] CLINE D B H. SAMORODNITSKY G. Convolution tails, product tails and domains of attraction [J]. Probab Theory Relat Fields, 1986, 72(4): 529-557.

[7] ASIMIT A V, BADESCU A L. Extremes on the discounted aggregate claims in a time-dependent risk model[J]. Scandinavian Actuarial Journal, 2010, 2010 (2): 93-104.

[8] LI J, TANG Q, WU R. Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model [J]. Advances in Applied Probability, 2010, 42: 1126-1146.

[9] CHEN Y, YUEN K C. Precise large deviations of aggregate claims in a size-dependent renewal risk model [J]. Insurance: Mathematics and Economics, 2012, 51 (2): 457-461.

[10] CLINE D B H, SAMORODNITSKY G. Subexponentiality of the product of the independent random variables[J]. Stoch Process Their Appl, 1994, 49(1): 75-98.

[11] TANG Q, TSITSIASHVILI G. Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks [J]. Stoch Process Their Appl, 2003, 108 (2): 299-325.

[12] GREY D R. Regular variation in the tail behaviour of solutions of random difference equations[J]. Annals of Applied Probability, 1994, 4: 169-183.

[13] GOLDIE C M, GRÜBEL R. Perpetuities with thin tails [J]. Adv Appl Probab, 1996, 28(2): 463-480.

[14] HITCZENKO P, WESOLOWSKI J. Perpetuities with thin tails revisited[J]. Ann Appl Probab, 2009, 19 (6): 2080-2101.

[15] YANG Y, WANG Y. Tail behavior of the product of two dependent random variables with applications to risk theory[J]. Extremes, 2013, 16(1): 55-74.

[16] QU Z, CHEN Y. Approximations of the tail probability of the product of dependent extremal random variables and applications [J]. Insurance: Mathematics and Economics, 2013, 53(1): 169-178.

[17] CHEN Y. The finite-time ruin probability with dependent insurance and financial risks[J]. Journal of Applied Probability, 2011, 48(4): 1035-1048.

[18] PAKES A G. Convolution equivalence and infinite divisibility[J]. J Appl Probab, 2004, 41(2): 407-424.