第45卷第1期 中国科学技术大学学报 Vol. 45, No. 1
2015年1月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Jan. 2015
Article ID:0253-2778(2015)01-0034-06

MacWilliams identities of linear codes over $M_{n\times s}(R_k)$ with respect to RT metric

YAO Ting, SHI Minjia, LIU Yan

(*School of Mathematical Sciences of Anhui University. Hefei 230601, China*)

Abstract: The definitions of the Lee complete ρ weight enumerator and the exact complete ρ weight enumerator over $M_{n\times s}(R_k)(u_i^2=0, u_iu_j=u_ju_i)$ were given, and the MacWilliams identities with respect to the RT metric for these two weight enumerators of linear codes over $M_{n\times s}(R_k)$ were obtained, respectively. Finally, two examples were presented to illustrate our obtained results.

Key words: RT metric; weight enumerator; MacWilliams identity; dual code

CLC number: TN911.22 **Document code**: A doi: 10.3969/j.issn.0253-2778.2015.01.006

AMS Subject Classification (2000): Primary 94B05; Secondary 94B99

Citation: Yao Ting, Shi Minjia, Liu Yan. MacWilliams identities of linear codes over $M_{n\times s}(R_k)$ with respect to RT metric[J]. Journal of University of Science and Technology of China, 2015,45(1):34-39.

环 $M_{n\times s}(R_k)$ 上线性码关于 RT 距离的 MacWilliams 恒等式

姚 婷,施敏加,刘 艳

(安徽大学数学科学学院,安徽合肥 230601)

摘要:定义了环 $M_{n\times s}(R_k)(u_i^2=0, u_iu_j=u_ju_i)$上线性码的 Lee 完全 ρ 重量计数器和精确完全 ρ 重量计数器,并给出了环 $M_{n\times s}(R_k)$上线性码关于这两种重量计数器的 MacWilliams 恒等式.最后给出两个例子对所得的结果进行了验证.

关键词:RT 距离;重量计数器;MacWilliams 恒等式;对偶码

0 Introduction

Error correcting codes have been extensively used in the field of computer security and communication, and the MacWilliams identity plays an important role in the theoretical study of error correcting codes. Therefore, many literatures studied MacWilliams identities over different alphabets. It is worth mentioning that Xu has contributed to MacWilliams identities with respect to RT(ρ) weight enumerator over different rings in Refs. [1-3]. In Ref. [4], Rosenbloom has studied RT(ρ) weight of codes over finite fields. A variety of weight enumerators and MacWilliams identities

Received: 2014-06-05; **Revised**: 2014-07-29

Foundation item: Supported by NNSF of China (61202068), Talents youth Fund of Anhui Province Universities (2012SQRL020ZD).

Biography: YAO Ting, female, born in 1989, master candidate. Research field: Algebraic codingtheory. E-mail: yaoting_1649@163.com

Corresponding author: SHI Minjia, PhD/associate professor. E-mail: smjwcl.good@163.com

for linear codes over $F_2+vF_2+v^2F_2$ were obtained in Ref. [5]. Siap investigated the MacWilliams identities with respect to complete weight enumerators over different alphabets in Refs. [6-8]. Liu etc. discussed the exact complete ρ weight enumerators over $M_{n\times s}(F_p+uF_p+vF_p+uvF_p)$ (F_p be a finite field with p elements) in Ref. [9].

Our aim in this article is to study the Lee complete ρ weight enumerator and the exact complete ρ weight enumerator over $M_{n\times s}(R_k)$, where $R_k=F_2[u_1,u_2,\cdots,u_k]/(u_i^2=0, u_iu_j=u_ju_i)$ and determine the MacWilliams identities with respect to these two weight enumerators. For more information on ring R_k, see Ref. [10]. Both the two weight enumerators are superior to complete ρ weight enumerator, and contain more information of codewords, which may play an important role in encoding and decoding linear codes over $M_{n\times s}(R_k)$. It is worth mentioning that Ref. [1] and Ref. [9] are special cases of this article.

1 Preliminary

Let $R=R_k=F_2[u_1,u_2,\cdots,u_k]/(u_i^2=0, u_iu_j=u_ju_i)=\{\sum_{A\subseteq\{1,\cdots,k\}}c_Au_A \mid c_A\in F_2\}$, where $u_A=\prod_{i\in A}u_i$, $u_i^2=0, u_iu_j=u_ju_i$, F_2 be the binary field. Traditionally, we write $u_\varnothing=1$. Let $M_{n\times s}(R)$ denote the set of all $n\times s$ matrices over R. Let $p=(p_0,p_1,\cdots,p_{s-1})\in M_{1\times s}(R)$. Then the RT weight of p is defined by

$$W_N(p)=\begin{cases}\max\{i: p_i\neq 0\}+1, & p\neq \mathbf{0}\\ 0, & p=\mathbf{0}\end{cases}.$$

The RT distance between p and q is defined as $\rho(p,q)=W_N(p-q)$, where $p, q\in M_{1\times s}(R)$. Furthermore, let $\Omega=(\omega_1,\omega_2,\cdots,\omega_n)^\top\in M_{n\times s}(R)$, where $\omega_i=(p_{i,0},p_{i,1},\cdots,p_{i,s-1})\in M_{1\times s}(R)$. Define the RT weight of Ω as $W_N(\Omega)=\sum_{i=1}^{n}W_N(\omega_i)$. The RT distance between Ω_1 and Ω_2 is $\rho(\Omega_1,\Omega_2)=W_N(\Omega_1-\Omega_2)$, where $\Omega_1, \Omega_2\in M_{n\times s}(R)$. Note that RT is a metric on R, and for $s=1$, the RT metric is just the usual Hamming metric.

A linear code C over $M_{n\times s}(R)$ is an R-submodule of $M_{n\times s}(R)$. The set

$$W_r(C)=|\{\Omega\in C \mid W_N(\Omega)=r\}|,$$

where $0\leqslant r\leqslant ns$ is called the ρ weight spectrum of a code C, and the ρ weight enumerator is defined by

$$W_C(z)=\sum_{r=0}^{ns}W_r(C)z^r=\sum_{\Omega\in C}z^{W_N(\Omega)}.$$

Let $\omega_1=(p_0,p_1,\cdots,p_{s-1})$ and $\omega_2=(q_0,q_1,\cdots,q_{s-1})$, where $\omega_1, \omega_2\in M_{1\times s}(R)$. Then the inner product of ω_1 and ω_2 is defined by $\langle\omega_1, \omega_2\rangle=\sum_{i=0}^{s-1}p_iq_{s-1-i}$, and this is extended to the inner product of Ω_1 and $\Omega_2\in M_{n\times s}(R)$ as $\langle\Omega_1, \Omega_2\rangle=\sum_{i=1}^{n}\langle\omega_i,\mu_i\rangle$, where $\Omega_1=(\omega_1,\omega_2,\cdots,\omega_n)^\top$, $\Omega_2=(\mu_1,\mu_2,\cdots,\mu_n)^\top\in M_{n\times s}(R)$, and $\omega_i=(p_{i,0},p_{i,1},\cdots,p_{i,s-1})$, $\mu_i=(q_{i,0},q_{i,1},\cdots,q_{i,s-1})\in M_{1\times s}(R)$.

Definition 1.1 The dual code of C is defined as $C^\perp=\{\Omega_2\in M_{n\times s}(R) \mid \langle\Omega_2, \Omega_1\rangle=0, \forall\,\Omega_1\in C\}$. $C^\perp$ is also a linear code over $M_{n\times s}(R)$.

The ring of $n\times s$ matrices over R can be identified with the ring of $n\times 1$ matrices which have polynomial entries. Before we give the definition of the complete weight enumerator, we first define the following map:

$$\Psi: M_{n\times s}(R)\to M_{n\times 1}(R[x]/(x^s))$$

$$P=(P_1,P_2,\cdots,P_n)^\top\to (P_1(x),P_2(x),\cdots,P_n(x))^\top$$

The map defined above is an R-module isomorphism.

Let $p(x)=p_0+p_1x+\cdots+p_{s-1}x^{s-1}\in R[x]/(x^s)$. Let the l^{th} $(0\leqslant l\leqslant s-1)$ coefficient of $p(x)$ be defined by $c_\ell(p(x))=p_\ell$, and the inner product of $p(x)$, $q(x)\in R[x]/(x^s)$ defined as $\langle p(x),q(x)\rangle=c_{s-1}(p(x)q(x))$. Let $P(x)=(P_1(x),P_2(x),\cdots,P_n(x))^\top$, $Q(x)=(Q_1(x),Q_2(x),\cdots,Q_n(x))^\top\in M_{n\times 1}(R[x]/(x^s))$, where $P_i(x)=p_{i,0}+p_{i,1}x+\cdots+p_{i,s-1}x^{s-1}$, $Q_i(x)=q_{i,0}+q_{i,1}x+\cdots+q_{i,s-1}x^{s-1}\in R[x]/(x^s)$, $1\leqslant i\leqslant n$. The inner product of $P(x)$ and $Q(x)$ defined above in terms of polynomials becomes $\langle P(x),Q(x)\rangle=\sum_{i=1}^{n}\langle P_i(x),Q_i(x)\rangle=\sum_{i=1}^{n}c_{s-1}(P_i(x)Q_i(x))$. The Hamming weight of an element $\alpha\in R$ is denoted by $w(\alpha)$ and it is equal to zero if $\alpha=0$, and 1 otherwise.

Definition 1.2 Let $Y_{ns}=(y_{1,0},\cdots,y_{1,s-1},\cdots,y_{n,0},\cdots,y_{n,s-1})$, $P=(p_{i,j})_{n\times s}\in M_{n\times s}(R)$, where $1\leqslant i\leqslant n$, $0\leqslant j\leqslant s-1$. Define the complete ρ weight enumerator of a code C over $M_{n\times s}(R)$ as $W_C(Y_{ns})=\sum_{P\in C} y_{1,0}^{w(p_{1,0})}\cdots y_{1,s-1}^{w(p_{1,s-1})}\cdots y_{n,0}^{w(p_{n,0})}\cdots y_{n,s-1}^{w(p_{n,s-1})}$. When $n=1$ and $s=m$, we define the complete ρ weight enumerator of a code C over R as

$$W_C(Y)=\sum_{p\in C} y_1^{w(p_0)}y_2^{w(p_1)}\cdots y_m^{w(p_{m-1})},$$

where $p=(p_0,p_1,\cdots,p_{m-1})\in R^m$, $Y=(y_1,y_2,\cdots,y_m)$.

Definition 1.3 Let $\sigma(\sum_{A\subseteq\{1,\cdots,k\}} c_A u_A)=c_k$, $\forall \sum_{A\subseteq\{1,\cdots,k\}} c_A u_A\in R$, where c_k denote a coefficient of $u_1u_2\cdots u_k$, $c_A\in F_2$. Define map χ: $R\to C^*$, $\chi(\alpha)=(-1)^{\sigma(\alpha)}$, $\forall\alpha\in R$. The χ is a characteristic of ring R.

2 Lee complete ρ weight enumerator

In this section, we will discuss the Lee complete ρ weight enumerator over $M_{n\times s}(R)$, which has the advantage of containing more information about the codewords of a code over $M_{n\times s}(R)$. Next we will prove a MacWilliams identity with respect to the Lee complete ρ weight enumerator over $M_{n\times s}(R)$.

Definition 2.1 Define Gray map Φ_k: $R^n\to F_2^{2^k n}$, $\Phi_k(a+bu_k)=(\Phi_{k-1}(b),\Phi_{k-1}(a)+\Phi_{k-1}(b))$, where $a+bu_k\in R_k^n$, $\forall a, b\in R_{k-1}^n$. Especially, when $k=1$, then $\Phi_1(a+bu)=(b,a+b)$. For $\forall\alpha\in R$, the Lee weight of α is defined as $w_L(\alpha)=w_H(\Phi_k(\alpha))$, where $w_H(\alpha)$ is a Hamming weight of α over F_2.

Definition 2.2 Let $Y=(y_1,y_2,\cdots,y_m)$ and $p=(p_0,p_1,\cdots,p_{m-1})$. Define the Lee complete ρ weight enumerator of a code C over R by

$$\mathrm{Lee}(Y)=\sum_{p\in C} y_1^{w_L(p_0)}y_2^{w_L(p_1)}\cdots y_m^{w_L(p_{m-1})} \tag{1}$$

Let $Y_{ns}=(y_{1,0},\cdots,y_{1,s-1},\cdots,y_{n,0},\cdots,y_{n,s-1})$, $P=(p_{i,j})_{n\times s}\in M_{n\times s}(R)$, where $1\leqslant i\leqslant n$, $0\leqslant j\leqslant s-1$. Define the Lee complete ρ weight enumerator of a code C over $M_{n\times s}(R)$ by

$$\mathrm{Lee}(Y_{ns})=\sum_{P\in C} y_{1,0}^{w_L(p_{1,0})}\cdots y_{1,s-1}^{w_L(p_{1,s-1})}\cdots y_{n,0}^{w_L(p_{n,0})}\cdots y_{n,s-1}^{w_L(p_{n,s-1})} \tag{2}$$

In particular, when $n=1$ and $s=m$ in this definition, by interchanging the subscripts if necessary, then Eq. (2) is none other than Eq. (1).

Similar to the proof of Lemma 2 in Ref. [3], we can get the following lemmas.

Lemma 2.1 Let C be a linear code over $M_{n\times s}(R)$ and $P(x)$, $Q(x)\in M_{n\times 1}(R[x]/(x^s))$. Then we have

$$\sum_{P(x)\in C}\chi(\langle P(x),Q(x)\rangle)=\begin{cases}0, & \text{if } Q\overline{\in} C^{\perp}\\ |C|, & \text{if } Q\in C^{\perp}\end{cases}.$$

Using Lemma 2.1, we obtain the following lemma which plays an important role in the main results of this article.

Lemma 2.2 Let C be a linear code over $M_{n\times s}(R)$, f: $M_{n\times 1}(R[x]/(x^s))\to\mathbb{C}[Y_{ns}]$, then

$$\sum_{Q(x)\in C^{\perp}} f(Q(x))=\frac{1}{|C|}\sum_{P(x)\in C}\hat{f}(P(x)),$$

where

$$\hat{f}(P(x))=\sum_{Q(x)\in M_{n\times 1}(R[x]/(x^s))}\chi(\langle P(x),Q(x)\rangle)\cdot f(Q(x)).$$

Lemma 2.3 Suppose β is fixed in R, then

$$\sum_{a\in R}\chi(\beta a)y^{w_L(a)}=(1+y)^{2^k-w_L(\beta)}(1-y)^{w_L(\beta)},\quad k\in N^+.$$

Proof We only prove the case when $k=2$ and the proof is similar when $k\geqslant 3$. By Definitions 1.3 and 2.1, we have

$$\sum_{a\in R}\chi(\beta a)y^{w_L(a)}=\begin{cases}(1+y)^4, & \beta=0\\ (1+y)^3(1-y), & \beta\in U\\ (1+y)^2(1-y)^2, & \beta\in V\\ (1+y)(1-y)^3, & \beta\in W\\ (1-y)^4, & \beta=u_1u_2\end{cases}$$
$$=(1+y)^{4-w_L(\beta)}(1-y)^{w_L(\beta)},$$

where $U=\{1,1+u_1,1+u_2,1+u_1+u_2+u_1u_2\}$, $V=\{u_1,u_2,u_1+u_2,u_1+u_1u_2,u_2+u_1u_2,u_1+u_2+u_1u_2\}$, $W=\{1+u_1u_2,1+u_1+u_2,1+u_1+u_1u_2,1+$

$u_2+u_1u_2\}$.

Next, we give a MacWilliams identity regarding the Lee complete ρ weight enumerator over $M_{n\times s}(R)$.

Theorem 2.1 Let C be a linear code over $M_{n\times s}(R)$, then

$$\sum_{Q(x)\in C^{\perp}} y_{1,0}^{w_L(q_{1,0})}\cdots y_{1,s-1}^{w_L(q_{1,s-1})}\cdots y_{n,0}^{w_L(q_{n,0})}\cdots y_{n,s-1}^{w_L(q_{n,s-1})}=$$

$$\frac{1}{|C|}\sum_{P(x)\in C}\prod_{i=1}^{n}\prod_{j=0}^{s-1}(1+y_{i,j})^{2^k}\times\prod_{t=1}^{n}\prod_{l=0}^{s-1}\left(\frac{1-y_{t,l}}{1+y_{t,l}}\right)^{w_L(p_{t,s-1-l})}.$$

Proof Let $f(Q(x))=\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{i,j}^{w_L(q_{i,j})}$, by Lemmas 2.1 and 2.2, we have

$$\hat{f}(P(x))=$$

$$\sum_{Q(x)\in M_{n\times 1}(R[x]/(x^s))}\chi(\langle P(x),Q(x)\rangle)\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{i,j}^{w_L(q_{i,j})}=$$

$$\sum_{q_{1,0}\in R}\chi(\langle P_1(x),q_{1,0}\rangle)y_{1,0}^{w_L(q_{1,0})}\cdots$$

$$\sum_{q_{1,s-1}\in R}\chi(\langle P_1(x),q_{1,s-1}x^{s-1}\rangle)y_{1,s-1}^{w_L(q_{1,s-1})}\cdots$$

$$\sum_{q_{n,0}\in R}\chi(\langle P_n(x),q_{n,0}\rangle)y_{n,0}^{w_L(q_{n,0})}\cdots$$

$$\sum_{q_{n,s-1}\in R}\chi(\langle P_n(x),q_{n,s-1}x^{s-1}\rangle)y_{n,s-1}^{w_L(q_{n,s-1})}=$$

$$(1+y_{1,0})^{2^k-w_L(p_{1,s-1})}(1-y_{1,0})^{w_L(p_{1,s-1})}\cdots$$

$$(1+y_{1,s-1})^{2^k-w_L(p_{1,0})}(1-y_{1,s-1})^{w_L(p_{1,0})}\cdots$$

$$(1+y_{n,0})^{2^k-w_L(p_{n,s-1})}(1-y_{n,0})^{w_L(p_{n,s-1})}\cdots$$

$$(1+y_{n,s-1})^{2^k-w_L(p_{n,0})}(1-y_{n,s-1})^{w_L(p_{n,0})}=$$

$$\prod_{i=1}^{n}\prod_{j=0}^{s-1}(1+y_{i,j})^{2^k-w_L(p_{i,s-1-j})}(1-y_{i,j})^{w_L(p_{i,s-1-j})}=$$

$$\prod_{i=1}^{n}\prod_{j=0}^{s-1}(1+y_{i,j})^{2^k}\prod_{t=1}^{n}\prod_{l=0}^{s-1}\left(\frac{1-y_{t,l}}{1+y_{t,l}}\right)^{w_L(p_{t,s-1-l})}.$$

We substitute it into Lemma 2.2 and the result is as follows.

Corollary 2.1 Let C be a linear code and $q(x)=q_0+q_1x+\cdots+q_{m-1}x^{m-1}$, $p(x)=p_0+p_1x+\cdots+p_{m-1}x^{m-1}$, where $q(x)$, $p(x)\in R[x]/(x^m)$, then

(Ⅰ) In Theorem 2.1, when $n=1$ and $s=m$, by properly interchanging the subscripts, Theorem 2.1 becomes

$$\sum_{q(x)\in C^{\perp}} y_1^{w_L(q_0)}y_2^{w_L(q_1)}\cdots y_m^{w_L(q_{m-1})}=\frac{1}{|C|}\sum_{p(x)\in C}\prod_{i=1}^{m}(1+y_i)^{2^k}\prod_{j=1}^{m}\left(\frac{1-y_j}{1+y_j}\right)^{w_L(p_{m-j})}\tag{3}$$

which is called a MacWilliams identity about the Lee complete ρ weight enumerator on a linear code C over R.

(Ⅱ) Let $s=1$ and $n=m$ be in Theorem 2.1, by interchanging the subscripts if necessary, Theorem 2.1 implies

$$\sum_{q(x)\in C^{\perp}} y_1^{w_L(q_0)}y_2^{w_L(q_1)}\cdots y_m^{w_L(q_{m-1})}=\frac{1}{|C|}\sum_{p(x)\in C}\prod_{i=1}^{m}(1+y_i)^{2^k}\prod_{j=1}^{m}\left(\frac{1-y_j}{1+y_j}\right)^{w_L(p_{j-1})}\tag{4}$$

a MacWilliams identity on the Lee weight enumerator of linear codes over R. Here, note that the inner product in Eq. (4) is just the ordinary Euclidean inner product.

3 Exact complete ρ weight enumerator

In this section, we will discuss exact complete ρ weight enumerator. We first define the exact weight and the exact complete ρ weight enumerator over $M_{n\times s}(R)$. Then, we give a MacWilliams identity about the exact complete ρ weight enumerator over $M_{n\times s}(R)$.

Definition 3.1 For $\forall\sum_{A\subseteq\{1,\cdots,k\}}c_Au_A\in R$, where $c_A\in F_2$, define exact weight as $w_e(\sum_{A\subseteq\{1,\cdots,k\}}c_Au_A)=\sum_{i=1}^{2^k}c_A2^{i-1}$.

Definition 3.2 Let $P=(p_{i,j})_{n\times s}\in M_{n\times s}(R)$, where $1\leqslant i\leqslant n$, $0\leqslant j\leqslant s-1$, $Y_{ns}=(y_{1,0},\cdots,y_{1,s-1},\cdots,y_{n,0},\cdots,y_{n,s-1})$. Define the exact complete ρ weight enumerator of a code C over $M_{n\times s}(R)$ by

$$E_C(Y_{ns})=\sum_{P\in C}y_{1,0}^{w_e(p_{1,0})}\cdots y_{1,s-1}^{w_e(p_{1,s-1})}\cdots y_{n,0}^{w_e(p_{n,0})}\cdots y_{n,s-1}^{w_e(p_{n,s-1})}\tag{5}$$

In particular, let $n=1$ and $s=m$ in Eq. (5), by properly interchanging the subscripts, we get the exact complete ρ weight enumerator of the code C over R, that is,

$$E_C(Y)=\sum_{p\in C} y_1^{w_e(p_0)}y_2^{w_e(p_1)}\cdots y_m^{w_e(p_{m-1})}.$$

Similar to the proof of Lemma 3 in Ref. [9], we have:

Lemma 3.1 Given $\beta\in R$, $P_i(x)=p_{i,0}+p_{i,1}x+\cdots+p_{i,n-1}x^{n-1}\in R[x]/(x^n)$, then

$$\sum_{a\in R}\chi(\beta a)y^{w_e(a)}=(1+\chi(\beta)y)(1+\chi(\beta u_1))y^2\cdots(1+\chi(\beta u_1u_2\cdots u_k)y^{2^{2^{k-1}}}),$$

where $k\in N^+$.

Theorem 3.1 Let C be a linear code over $M_{n\times s}(R)$, then

$$\sum_{Q(x)\in C^\perp} y_{1,0}^{w_e(q_{1,0})}\cdots y_{1,s-1}^{w_e(q_{1,s-1})}\cdots y_{n,0}^{w_e(q_{n,0})}\cdots y_{n,s-1}^{w_e(q_{n,s-1})}=$$

$$\frac{1}{|C|}\sum_{P(x)\in C}\prod_{t=1}^{n}\prod_{h=0}^{s-1}[1+\chi(p_{t,s-1-h})y_{t,h}]\cdot[1+\chi(p_{t,s-1-h}u_1)y_{t,h}^{2}]\cdots[1+\chi(p_{t,s-1-h}u_1u_2\cdots u_k)y_{t,h}^{2^{2^{k-1}}}].$$

Proof Let $f(Q(x))=f((Q_1(x),\cdots,Q_n(x))^{\top})=\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{i,j}^{w_e(q_{i,j})}$, by Lemmas 2.2 and 3.1, we have

$$\hat f(P(x))=\sum_{Q(x)\in M_{n\times1}(R[x]/(x^s))}\chi(\langle P(x),Q(x)\rangle)f(Q(x))=$$

$$\sum_{q_{1,0}\in R}\chi(\langle P_1(x),q_{1,0}\rangle)y_{1,0}^{w_e(q_{1,0})}\cdots\sum_{q_{1,s-1}\in R}\chi(\langle P_1(x),q_{1,s-1}x^{s-1}\rangle)y_{1,s-1}^{w_e(q_{1,s-1})}\cdots$$

$$\sum_{q_{n,0}\in R}\chi(\langle P_n(x),q_{n,0}\rangle)y_{n,0}^{w_e(q_{n,0})}\cdots\sum_{q_{n,s-1}\in R}\chi(\langle P_n(x),q_{n,s-1}x^{s-1}\rangle)y_{n,s-1}^{w_e(q_{n,s-1})}=$$

$$\prod_{h=0}^{s-1}[1+\chi(p_{1,h})y_{1,s-1-h}][1+\chi(p_{1,h}u_1)y_{1,s-1-h}^{2}]\cdots[1+\chi(p_{1,h}u_1u_2\cdots u_k)y_{1,s-1-h}^{2^{2^{k-1}}}]\cdots$$

$$\prod_{h=0}^{s-1}[1+\chi(p_{n,h})y_{n,s-1-h}][1+\chi(p_{n,h}u_1)y_{n,s-1-h}^{2}]\cdots[1+\chi(p_{n,h}u_1u_2\cdots u_k)y_{n,s-1-h}^{2^{2^{k-1}}}]=$$

$$\prod_{t=1}^{n}\prod_{h=0}^{s-1}[1+\chi(p_{t,s-1-h})y_{t,h}][1+\chi(p_{t,s-1-h}u_1)y_{t,h}^{2}]\cdots[1+\chi(p_{t,s-1-h}u_1u_2\cdots u_k)y_{t,h}^{2^{2^{k-1}}}],$$

the result then follows from applying Lemma 2.2.

Corollary 3.1 Let C be a linear code and $q(x)=q_0+q_1x+\cdots+q_{m-1}x^{m-1}$, $p(x)=p_0+p_1x+\cdots+p_{m-1}x^{m-1}\in R[x]/(x^m)$, then

(Ⅰ) when $n=1$ and $s=m$, we have

$$\sum_{q(x)\in C^\perp} y_1^{w_e(q_0)}y_2^{w_e(q_1)}\cdots y_m^{w_e(q_{m-1})}=$$

$$\frac{1}{|C|}\sum_{p(x)\in C}\prod_{t=1}^{m}[1+\chi(p_{m-t})y_t][1+\chi(p_{m-t}u_1)y_t^{2}]\cdots[1+\chi(p_{m-t}u_1u_2\cdots u_k)y_t^{2^{2^{k-1}}}],$$

which is called a MacWilliams identity about the exact complete ρ weight enumerator on a linear code C over R.

(Ⅱ) when $s=1$ and $n=m$, we obtain

$$\sum_{q(x)\in C^\perp} y_1^{w_e(q_0)}y_2^{w_e(q_1)}\cdots y_m^{w_e(q_{m-1})}=$$

$$\frac{1}{|C|}\sum_{p(x)\in C}\prod_{t=1}^{m}[1+\chi(p_{t-1})y_t][1+\chi(p_{t-1}u_1)y_t^{2}]\cdots[1+\chi(p_{t-1}u_1u_2\cdots u_k)y_t^{2^{2^{k-1}}}],$$

a MacWilliams identity about the exact weight enumerator of a linear code C over R.

Note The inner product in Eq. (2) is none other than the ordinary Euclidean inner product.

4 Application examples

In the previous sections, MacWilliams identities about Lee complete ρ weight enumerator and exact complete ρ weight enumerator are given. In this section, we mainly give some examples to illustrate the application of the main results.

Let C be a linear code over R, and $R_2=F_2+u_1F_2+u_2F_2+u_1u_2F_2$ $(u_1^2=u_2^2=0,\ u_1u_2=u_2u_1)$. C is generated by $\{(1,1+u_1u_2,u_1),(0,u_1u_2,u_2)\}$, and the dual code of C is generated by $\{(u_1,1+u_1u_2,1),(u_2,u_1u_2,0)\}$. By Definition 2.2, we have

$$\mathrm{Lee}_C(Y)=1+y_3^4+2y_2^4y_3^2+y_1^4y_2^4+y_1^4y_2^4y_3^4+2y_1^4y_3^2+6y_1^2y_2^2y_3^4+6y_1^2y_2^2+12y_1^2y_2^2y_3^2+8y_1y_2^3y_3^2+8y_1y_2y_3^2+8y_1^3y_2y_3^2+8y_1^3y_2^3y_3^2.$$

According to Corollary 2.1, we have

$\mathrm{Lee}_{C^{\perp}}(Y) =$

$$\frac{1}{|C|}\sum_{P\in C}\prod_{i=1}^{3}(1+y_i)^{2^2}\prod_{j=1}^{3}\left(\frac{1-y_j}{1+y_j}\right)^{w_L(p_{3-j})} =$$

$$1+y_1^4+2y_1^2y_2^4+y_2^4y_3^4+y_1^4y_2^4y_3^4+2y_1^2y_3^4+6y_1^4y_2^2y_2^2+6y_2^2y_3^2+12y_1^2y_2^2y_3^2+8y_1^2y_2^3y_3+8y_1^2y_2y_3+8y_1^2y_2y_3^3+8y_1^2y_2^3y_3^3.$$

By Definition 3.2, we get

$E_C(Y) =$

$$1+y_1^2y_2^2+y_3^8+\cdots+y_1^{15}y_2^{15}y_3^6+y_1^{15}y_2^{15}y_3^{14}.$$

According to Corollary 3.5, we have

$E_{C^{\perp}}(Y) =$

$$\frac{1}{|C|}\sum_{P\in C}\prod_{t=1}^{2}[1+\chi(p_{2-t})y_t][1+\chi(p_{2-t}u_1)y_t^2]\cdot[1+\chi(p_{2-t}u_2)y_t^4][1+\chi(p_{2-t}u_1u_2)y_t^8] =$$

$$1+y_2^3y_3^2+y_1^8+\cdots+y_1^6y_2^{15}y_3^{15}+y_1^{14}y_2^{15}y_3^{15}.$$

On the other hand, all codewords of the code can be listed, by Definitions 2.2 and 3.2. It is easy to verify that the results are correct.

5 Conclusion

This article gives MacWilliams identities about the Lee complete ρ weight enumerator and the exact complete ρ weight enumerator of linear codes over $M_{n\times s}(R_k)$ (see Theorem 2.1 and Theorem 3.1). Note that the case where $k=2$ in Theorem 3.1 is identified with the case where $p=2$ in Theorem 1 in Ref. [9], and Theorem 1.7 in Ref. [1] is none other than the case of this article (see Theorem 2.1) when $k=1$. Namely, Refs. [1] and [9] are special cases of this article.

References

[1] Xu H Q, Zhu S X. MacWilliams identities of linear codes over ring $M_{n\times s}(F_2+uF_2)$ with respect to RT metric [J]. Journal of University of Science and Technology of China, 2008, 38(9):1 075-1 080.
许和乾，朱士信. 环 $M_{n\times s}(F_2+uF_2)$上线性码关于 RT 距离的 MacWilliams 恒等式[J]. 中国科学技术大学学报，2008，38(9)：1 075-1 080.

[2] Xu H Q, Du W. A MacWilliams identity respecting to ρ metric[J]. Computer Engineering, 2012, 38(19): 122-125.
许和乾，杜炜. 关于 ρ 度量的一个 MacWilliam 恒等式[J]. 计算机工程，2012，38(19)：122-125.

[3] Zhu S X, Xu H Q, Shi M J. MacWilliams identity with respect to RT metric over ring Z_4 [J]. Acta Electronica Sinica, 2009, 37(5): 1 115-1 118.
朱士信，许和乾，施敏加. 环 Z_4 上线性码关于 RT 距离的 MacWilliams 恒等式[J]. 电子学报，2009，37(5)：1 115-1 118.

[4] Rosenbloom M Y, Tsfasman M A. Codes for the m-metric [J]. Journal of Problems Information Transmission, 1997, 33(1): 45-52.

[5] Shi M J, Solé P, Wu B. Cyclic codes and the weight enumerator of linear codes over $F_2+vF_2+v^2F_2$ [J]. Applied and Computational Mathematics, 2013, 12(2):247-255.

[6] Siap I. The complete weight enumerator for codes over $M_{n\times s}(F_q)$ [J]. Lecture Notes on Computer Sciences, 2001, 2260(8): 20-26.

[7] Siap I. A MacWilliams type identity [J]. Turkey Journal of Mathematics, 2002, 26(4): 465-473.

[8] Siap I, Ozen M. The complete weight enumerator for codes over $M_{n\times s}(R)$[J]. Applied Mathematics Letters, 2004, 17(1): 65-69.

[9] Liu Y, Shi M J. The MacWilliams identity of linear codes with respect to RT metric over $M_{n\times s}(F_p+uF_p+vF_p+uvF_p)$ with respect to RT metric [C]// The International Conference on Computers and Information Processing Technologies. Shanghai, China: IEEE Press, 2014.

[10] Dougherty S T, Yildiz B, Karadeniz S. Codes over R_k, gray maps and their binary images [J]. Finite Fields and Their Applications, 2011, 17(3): 205-219.