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On the quasi Gauss map for a compact sub-manifold in Euclidean space

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Abstract: Let σ be the quasi Gauss map of a compact and oriented n-dimensional isometric immersion sub-manifold M^n in the $(n+p)$-dimensional Euclid space R^{n+p}. Denote by ξ the unit mean curvature vector field to M^n and denote by H_i the i-mean curvature along the direction ξ. Assume that $H_i>0$, $i=1,2,\cdots,r$ for some integer r ($1\leqslant r\leqslant n$-1) and H_r is a constant. By applying an integral formula recently given by themselves, it is proven that if the image $\sigma(M^n)$ lies within an open n-dimension unit semi sphere S^n_+ then M^n must be totally quasi umbilical. This result generalizes a relevant theorem on hypersurfaces in Euclid space.

Key words: Euclid space; compact sub-manifold without boundary; mean curvature vector field; quasi Gauss map; i-mean curvature; totally quasi umbilical

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欧氏空间中紧致子流形的拟高斯映照

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摘要:令 M^n 为 $(n+p)$ 维欧氏空间 R^{n+p} 中 n 维定向的紧致无边子流形,而 σ 为 M^n 的拟高斯映照. 用 ξ 表示 M^n 的单位平均曲率向量场,而 H_i 表示 M^n 沿 ξ 方向的 i-平均曲率. 假设对某个整数 $r(1\leqslant r\leqslant n$-1) 而言有 $H_i>0$, $i=1,2,\cdots,r$ 而且 H_r 为常数. 利用作者自己最近得到的一个积分公式,证明了:如果 $\sigma(M^n)$ 落在一个开的 n 维半球面 S^n_+ 中,则 M^n 必全拟脐. 结果推广了有关欧氏空间中超曲面的一个相关定理.

关键词:欧氏空间; 紧致无边子流形; 平均曲率向量场; 拟高斯映照; i-平均曲率;全拟脐

0 Introduction

As well known the study of hyper-surfaces and sub-manifolds in Euclid space is one of the most important tasks in differential geometry.

For the research of describing properties of hyper-surfaces by using i-mean curvature and Gauss map, reference [1] attained the following

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Theorem 0.1.

Theorem 0.1[1] Let M be a compact and oriented hyper-surface without boundary in the $(n+1)$-dimensional Euclid space R^{n+1} and let M be convex. Assume that the r-mean curvature H_r is a constant for some integer $r\,(1\leqslant r\leqslant n-1)$. If the Gauss map for M is a topological homomorphism on to the n-dimensional unit sphere S^n, then M must be totally umbilical,

In this paper we study the totally quasi umbilical property of an n-dimensional compact sub-manifold M^n in the $(n+p)$-dimensional Euclid space R^{n+p}.

At first we define the quasi Gauss map σ for M^n and then, by applying an integral formula recently given in Ref. [2], we attain the following Theorem 0.2.

Theorem 0.2 Let R^{n+p} be the $(n+p)$-dimensional Euclid space and M^n be a compact and oriented n-dimension isometric immersion sub-manifold in R^{n+p}. Denote by ξ the unit mean curvature vector field to M^n and by H_i the i-mean curvature along the direction ξ, and denote by σ the quasi Gauss map for M^n. Assume that $H_i>0$, $i=1,2,\cdots,r$ for some integer r $(1\leqslant r\leqslant n-1)$ and H_r is a constant. If the image $\sigma(M^n)$ lies in an open n-dimension semi-sphere S^n_+, then M^n must be totally quasi umbilical.

Remark 0.1 In Theorem 0.1, the hyper-surface M is assumed to be convex and H_r is a constant, then actually we have $H_i>0$, $i=1,2,\cdots,r$. So Theorem 0.2 generalizes Theorem 0.1.

1 Preparation

Let R^{n+p} be the $(n+p)$-dimensional Euclid space and let $M^n=(M^n,g)$ be an n-dimensional smooth Riemann manifold.

Denote by $\varphi: M^n\to R^{n+p}$ a smooth immersion mapping. If equation

$$g=\varphi^*(\langle,\rangle)$$

holds everywhere on M^n, then M^n is said to be an isometric immersion sub-manifold. Here $\langle,\rangle$ is the Euclid inner product of R^{n+p}.

Let ξ be the unit mean curvature vector field to M^n (see Ref. [2]).

Definition 1.1 Let $\varphi: M^n\to R^{n+p}$ be an isometric immersion sub-manifold and ξ be the unit mean curvature vector field to M^n and let S^n be the n-dimensional standard unit sphere. Then the mapping

$$\sigma: M^n\to S^n,\, x\in M^n\mapsto\sigma(x)=\xi(x)\in S^n$$

is called the quasi Gauss map for M^n.

Definition 1.2 Let $\varphi: M^n\to R^{n+p}$ be an isometric immersion sub-manifold and ξ be the unit mean curvature vector field to M^n. Denote by λ_1, $\lambda_2,\cdots,\lambda_n$ the principal curvature functions of M^n along the direction ξ. If equation

$$\lambda_1=\lambda_2=\cdots=\lambda_n$$

holds at a point $x\in M^n$, then x is called a quasi umbilical point. If every point in M^n is quasi umbilical, then M^n is said to be a totally quasi umbilical sub-manifold.

Lemma 1.1[2] Let $\varphi: M^n\to R^{n+p}$ be a compact and oriented n-dimension isometric immersion sub-manifold without boundary. Then the following integral formulas hold.

$$\int_M (H_k+H_{k+1}\langle\varphi,\xi\rangle)\,\mathrm{d}V=0,\ k=0,1,2,\cdots,n-1.$$

Here ξ is the unit mean curvature vector field to M^n, H_k is the k-mean curvature of M^n along the direction ξ and $\langle\varphi,\xi\rangle$ is the Euclid inner product in R^{n+p}, and dV is the n-dimensional Riemann volume form of M^n.

2 Proof of Theorem 0.2

Theorem 0.2 Let R^{n+p} be the $(n+p)$-dimensional Euclid space and M^n be a compact and oriented n-dimension isometric immersion sub-manifold in R^{n+p}. Denote by ξ the unit mean curvature vector field to M^n and by H_i the i-mean curvature along the direction ξ, and denote by σ the quasi Gauss map for M^n. Assume that $H_i>0$, $i=1,2,\cdots,r$ for some integer r $(1\leqslant r\leqslant n-1)$ and H_r is a constant. If the image $\sigma(M^n)$ lies in an open n-dimension semi-sphere S^n_+, then M^n must be totally quasi umbilical.

Proof From Lemma 1.1 we have

$$\int_M (1 + H_1 \langle \varphi, \xi \rangle) \, dV = 0 \quad (1)$$

$$\int_M (H_r + H_{r+1} \langle \varphi, \xi \rangle) \, dV = 0 \quad (2)$$

Since H_r is a constant, from Eq.(1) we have

$$\int_M (H_r + H_1 H_r \langle \varphi, \xi \rangle) \, dV = 0 \quad (3)$$

From Eqs.(2) and (3) we have

$$\int_M (H_1 H_r - H_{r+1}) \langle \varphi, \xi \rangle \, dV = 0 \quad (4)$$

Now we recall a famous inequality (see Refs. [1,3,4] and prescribe $H_0 \equiv 1, H_{n+1} \equiv 0$)

$$H_i^2 \geqslant H_{i-1} H_{i+1}, \; i = 1, 2, \cdots, n \quad (5)$$

Since we assume that $H_i > 0$, $i = 1, 2, \cdots, r$ and from Eq.(5) we have

$$H_1 \geqslant \frac{H_2}{H_1} \geqslant \cdots \geqslant \frac{H_r}{H_{r-1}} \geqslant \frac{H_{r+1}}{H_r}, \; \forall x \in M^n \quad (6)$$

From Eq.(6) we have

$$(H_1 H_r - H_{r+1}) \geqslant 0, \; \forall x \in M^n \quad (7)$$

Noticing that M^n is compact we can, by translation, assume that M^n lies inside an n-dimensional cone with center at the original point and with intersectional angle small enough. At the same time we assume that the quasi Gauss image $\sigma(M^n)$ lies inside an open semi-sphere of n-dimension. And so we have

$$\langle \varphi, \xi \rangle = \langle \varphi(x), \xi(x) \rangle > 0, \; \forall x \in M^n \quad (8)$$

Combining Eqs.(4), (7) and (8) we have

$$H_1 H_r - H_{r+1} \equiv 0 \quad (9)$$

And so by combining Eqs.(6) and (9) we have

$$H_r^2 \equiv H_{r-1} H_{r+1} \quad (10)$$

We notice that inequality (6) attains its equality sign at and only at a quasi-umbilical point (see Refs.[1,3,4]).

Finally from Eq.(10) we already finish the proof.

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