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Multiplicity of solutions to elliptic equations with exponential nonlinearities

XUE Yimin, CHEN Shouting
(*School of Mathematics and Physical Science, Xuzhou University of Technology, Xuzhou, Jiangsu* 221018, *China*)

Abstract: The multiplicity of positive solutions to quasi-linear elliptic equations with exponential nonlinearities is obtained through a singular Trudinger-Moser inequality, which is due to Ref. [21], the mountain-pass theorem without the Palais-Smale condition and the Ekeland's variational principle. In particular, for the proof of our main results, we follow the lines of Refs. [11,15].

Key words: Singular Trudinger-Moser inequality; Mountain-pass theorem; exponential growth; multiple solutions
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含指数增长非线性项的椭圆方程的多解性

薛益民,陈守婷
(徐州工程学院数学与物理科学学院,江苏徐州 221018)

摘要:利用 Li 的一个奇异 Trudinger-Moser 不等式、没有 Palais-Smale 条件的山路引理和 Ekeland 的变分原理,借鉴文献[11,15]的方法证明了一类含指数增长非线性项的椭圆形方程的多解性.

关键词:奇异 Trudinger-Moser 不等式;山路引理;指数增长;多解

0 Introduction and main results

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ ($N\geqslant 2$), $W_0^{1,p}(\Omega)$ ($p\geqslant 1$) be the usual Sobolev space, a completion of $C_0^1(\Omega)$ under the norm

$$\| u \|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)^{1/p}.$$

Then the Sobolev embedding gives

$$W_0^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{Np}{N-p}}(\Omega), & \text{when } 1\leqslant p < N, \\ C^{1-\frac{N}{p}}(\overline{\Omega}), & \text{when } p > N. \end{cases}$$

When $p=N$, it is known as the Trudinger-Moser embedding[1-5], namely

$$\sup_{u\in W_0^{1,N}(\Omega),\ \|u\|_{W_0^{1,N}(\Omega)}\leqslant 1}\int_\Omega e^{\alpha|u|^{\frac{N}{N-1}}}\mathrm{d}x < \infty,\quad \forall\, \alpha\leqslant \alpha_N = N\omega_{N-1}^{1/(N-1)} \tag{1}$$

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Biography: XUE Yimin(corresponding author), male, born in 1977, master/lecturer. Research field: Differential equations and their applications. E-mail: xueym@xzit.edu.cn

where ω_{N-1} denotes the area of the unit sphere in $\mathbb{R}^N$. Moreover, α_N is the best constant in the sense that if $\alpha > \alpha_N$, all integrals in Eq. (1) are still finite, but the supremum is infinite. In Ref. [6], Eq. (1) was extended by Adimurthi and Sandeep to a singular Trudinger-Moser inequality, namely

$$\sup_{u\in W_0^{1,N}(\Omega),\ \|u\|_{W_0^{1,N}(\Omega)}\leqslant 1}\int_\Omega \frac{e^{\alpha_N(1-\beta)|u|^{\frac{N}{N-1}}}}{|x|^{N\beta}}\mathrm{d}x<\infty,\quad \forall\, 0\leqslant\beta<1 \tag{2}$$

For unbounded domain, in particular for $\mathbb{R}^N$, it was proved by Refs. [7-10]

$$\sup_{u\in W^{1,N}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}(|\nabla u|^N+|u|^N)\mathrm{d}x\leqslant 1}\int_{\mathbb{R}^N}\left(e^{\alpha_N|u|^{\frac{N}{N-1}}}-\sum_{k=0}^{N-2}\frac{\alpha_N^k\,|u|^{\frac{kN}{N-1}}}{k!}\right)\mathrm{d}x<\infty \tag{3}$$

An analog of Eq. (2) in $\mathbb{R}^N$ was due to Adimurthi and Yang[11], namely for any β with $0\leqslant\beta<N$,

$$\sup_{u\in W^{1,N}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}(|\nabla u|^N+|u|^N)\mathrm{d}x\leqslant 1}\int_{\mathbb{R}^N}\left(e^{\alpha_N(1-\beta)|u|^{\frac{N}{N-1}}}-\sum_{k=0}^{N-2}\frac{\alpha_N^k(1-\beta)^k\,|u|^{\frac{kN}{N-1}}}{k!}\right)\frac{1}{|x|^\beta}\mathrm{d}x<\infty \tag{4}$$

Obviously Eq. (4) is reduced to Eq. (3) in the case $\beta=0$. In Refs. [8,11-16], Eqs. (3) and (4) were applied to obtain the existence results for quasi-linear equations of the form

$$-\mathrm{div}(|\nabla u|^{N-2}\nabla u)+V(x)|u|^{N-2}u=\frac{f(x,u)}{|x|^\beta} \tag{5}$$

for certain potential $V:\mathbb{R}^N\to\mathbb{R}$ and exponential nonlinearity $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$. For similar problems, we refer the readers to Refs. [17-18].

Let E be the function space defined by

$$E=\left\{u\in W^{1,N}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)|u|^N\mathrm{d}x<\infty\right\}.$$

Motivated by recent works of Refs. [19-21], we consider the equation

$$-\mathrm{div}(|\nabla u|^{N-2}\nabla u)+V(x)|u|^{N-2}u-\alpha\|u\|^{N-p}_{L^p(\mathbb{R}^N)}|u|^{p-2}u=\frac{f(x,u)}{|x|^\beta}+\epsilon h(x) \tag{6}$$

where $p\geqslant N$, $0<\beta<N$, ϵ is a positive real constant, $h:\mathbb{R}^N\to\mathbb{R}$ is a function in E^*, the dual of E, V has positive lower bound in $\mathbb{R}^N$, $f(x,u)$ has exponential growth like $e^{\alpha u^{N/(N-1)}}$ as $|u|\to\infty$, and α is some positive constant. For convenience, we denote a function $\zeta:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ as

$$\zeta(N,s)=e^s-\sum_{k=0}^{N-2}\frac{s^k}{k!}=\sum_{k=N-1}^{\infty}\frac{s^k}{k!} \tag{7}$$

Throughout this paper, $V(x)$ and $f(x,s)$ are assumed to satisfy

(H_1) $V(x)\geqslant V_0>0$ in $\mathbb{R}^N$ for some constant $V_0>0$.

(H_2) $\dfrac{1}{V(x)}\in L^{\frac{1}{N-1}}(\mathbb{R}^N)$.

(H_3) There exist positive real constants α_0, a_1, a_2 such that

$$|f(x,s)|\leqslant a_1 s^{N-1}+a_2\zeta(N,\alpha_0 s^{\frac{N}{N-1}}),\quad \forall(x,s)\in\mathbb{R}^N\times\mathbb{R}^+.$$

(H_4) There exist $\mu>N$ such that

$$0<\mu F(x,s)\equiv\mu\int_0^s f(x,t)\mathrm{d}t\leqslant sf(x,s).$$

(H_5) There exist positive real constants R_0, M_0 such that

$$F(x,s)\leqslant M_0 f(x,s),\quad \forall x\in\mathbb{R}^N,\ s\geqslant R_0.$$

According to Refs. [11,14-15], we assume throughout this paper

$$f(x,s)\equiv 0,\quad \forall(x,s)\in\mathbb{R}^N\times(-\infty,0) \tag{8}$$

It follows from (H_1) that E is a reflexive Banach space endowed with a norm

$$\|u\|_{E,\alpha}=\left(\int_{\mathbb{R}^N}(|\nabla u|^N+V|u|^N)\mathrm{d}x-\alpha\|u\|^N_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{N}} \tag{9}$$

where $p\geqslant N$ and α satisfies

$$0\leqslant\alpha<\lambda_{N,p}:=\inf_{u\in E,u\not\equiv 0}\frac{\int_{\mathbb{R}^N}(|\nabla u|^N+V_0|u|^N)\mathrm{d}x}{\left(\int_{\mathbb{R}^N}|u|^p\mathrm{d}x\right)^{N/p}} \tag{10}$$

According to Ref. [21], under the assumptions of (H_1) and Eq. (10), there holds an analog of Eq. (4),

$$\sup_{u\in E,\ \|u\|_{E,\alpha}\leqslant 1}\int_{\mathbb{R}^N}\left(e^{\alpha_N(1-\beta)|u|^{\frac{N}{N-1}}}-\sum_{k=0}^{N-2}\frac{\alpha_N^k(1-\beta)^k\,|u|^{\frac{kN}{N-1}}}{k!}\right)\frac{1}{|x|^{\beta}}\mathrm{d}x<\infty \tag{11}$$

Now for any β, $0<\beta<N$, we define

$$\lambda_\beta=\inf_{u\in E,u\not\equiv 0}\frac{\|u\|_{E,\alpha}^N}{\int_{\mathbb{R}^N}\frac{|u|^N}{|x|^{\beta}}\mathrm{d}x} \tag{12}$$

It is clear that $\lambda_\beta>0$. Denote $K(r)=\sup_{|x|\leqslant r}V(x)$ and

$$M=\inf_{r>0}\frac{(N-\beta)^N}{\alpha_0^{N-1}r^{N-1}}e^{(N-\beta)\frac{(N-2)!}{N^N}K(r)r^N} \tag{13}$$

where α_0 is given by (H_3). Owing to $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ and (H_1), one can obtain that $K(r)>0$ is continuous and M can be attained by some $r>0$.

Now our main results can be stated as follows:

Theorem 0.1 Let $0<\beta<N$ be fixed and M be given as in Eq. (13). Suppose that $V\in C(\mathbb{R}^N,\mathbb{R})$ and $f\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$ satisfy (H_1)~(H_5), h belongs to E^* and $0\leqslant h\not\equiv 0$. Moreover, the following two hypotheses are satisfied:

(H_6) $\limsup_{s\to 0^+}\frac{N|F(x,s)|}{s^N}<\lambda_\beta$ holds uniformly for $x\in\mathbb{R}^N$;

(H_7) $\liminf_{s\to+\infty}sf(x,s)e^{\alpha_0 s^{N/(N-1)}}=\beta_0>M$ holds uniformly for $x\in\mathbb{R}^N$.

Then for any α satisfying Eq. (10), there exists $\epsilon_0>0$ such that if $0<\epsilon<\epsilon_0$, then the Eq. (6) has two distinct positive weak solutions.

To prove Theorem 0.1, we employ the singular Trudinger-Moser inequality (11) instead of Eq. (4). This is the essential difference between our main results and that of Yang[15]. The remaining part of this paper is organized as follows: In Section 1, we give several technical lemmas. In Section 2, the variational frameworks related to equations (6) is established. Finally, we prove Theorem 0.1 in Section 3.

1 Preliminaries

In this section, we give some preliminary analysis for our use later. One is a Lions' type lemma, the other is an embedding of the space E. Let $0<\beta<N$ be fixed.

Lemma 1.1 Let a sequence $\{w_k\}\subset E$, such that $\|w_k\|_{E,\alpha}=1$, $w_k\rightharpoonup w_0$ weakly in E, $w_k(x)\to w_0(x)$ and $\nabla w_k(x)\to\nabla w_0(x)$ for almost every $x\in\mathbb{R}^N$. Then, for any $0<p<1/(1-\|w_0\|_{E,\alpha}^N)^{1/(N-1)}$, we have

$$\sup_k\int_{\mathbb{R}^N}\frac{\zeta(N,\alpha_N(1-\beta/N)p\,|w_k|^{\frac{N}{N-1}})}{|x|^{\beta}}\mathrm{d}x<\infty.$$

Proof Firstly, for any $\epsilon>0$, the Young inequality yields

$$|w_k|^{\frac{N}{N-1}}=|w_k-w_0+w_0|^{\frac{N}{N-1}}\leqslant (1+\epsilon)\,|w_k-w_0|^{\frac{N}{N-1}}+c(\epsilon)\,|w_0|^{\frac{N}{N-1}} \tag{14}$$

For any positive real constants μ, υ with $1/\mu+1/\upsilon=1$, in view of Eq. (14) and (Ref. [15], Lemma 2.2), we have

$$\begin{aligned}&\zeta(N,\alpha_N(1-\beta/N)p\,|w_k|^{\frac{N}{N-1}})\leqslant\\&\frac{1}{\mu}\zeta\left(N,\mu(1+\epsilon)\alpha_N\left(1-\frac{\beta}{N}\right)p\,|w_k-w_0|^{\frac{N}{N-1}}\right)+\\&\frac{1}{\upsilon}\zeta\left(N,\upsilon c(\epsilon)\alpha_N\left(1-\frac{\beta}{N}\right)p\,|w_0|^{\frac{N}{N-1}}\right)\leqslant\\&\zeta(N,\mu(1+\epsilon)\alpha_N\left(1-\frac{\beta}{N}\right)p\,\|w_k-w_0\|_{E,\alpha}^{\frac{N}{N-1}}\cdot\\&\left(\frac{|w_k-w_0|}{\|w_k-w_0\|_{E,\alpha}}\right)^{\frac{N}{N-1}})+\\&\zeta\left(N,\upsilon c(\epsilon)\alpha_N\left(1-\frac{\beta}{N}\right)p\,|w_0|^{\frac{N}{N-1}}\right).\end{aligned}$$

By using Eq. (9) and (Ref. [22], Theorem 0.1), one gets

$$\lim_{k\to\infty}\|w_k\|_{E,\alpha}^N-\|w_k-w_0\|_{E,\alpha}^N=\|w_0\|_{E,\alpha}^N.$$

Since $\|w_k\|_{E,\alpha}=1$, it is easy to see that

$$\|w_k-w_0\|_{E,\alpha}^N=1-\|w_0\|_{E,\alpha}^N+o_k(1),$$

where $o_k(1)\to 0$ as $k\to\infty$. Therefore for any $0<p<1/(1-\|w_0\|_{E,\alpha}^N)^{1/(N-1)}$, we can fix a small positive real number ϵ and a positive real number μ close enough to 1 such that

$$0<p<\frac{1}{\mu(1+\epsilon)(1-\|w_0\|_{E,\alpha}^N+o_k(1))^{1/\{(N-1)\}}}$$

and that

$$\mu(1+\epsilon)\alpha_N(1-\beta/N)p\,\|w_k-w_0\|_{E,\alpha}<\alpha_N(1-\beta/N).$$

This together with Eq. (11) gives the desired result.

Lemma 1.2 Suppose that $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ and (H_1), (H_2) hold. Then, for any $q\geqslant 1$ the embedding $E\hookrightarrow L^q(\mathbb{R}^N)$ is compact.

Proof Firstly, we prove that the embedding $E\hookrightarrow L^q(\mathbb{R}^N)$ is continuous. By (H_1) and the Sobolev embedding theorem, one can obtain that the embedding $E\hookrightarrow W^{1,N}(\mathbb{R}^N)\hookrightarrow L^q(\mathbb{R}^N)$ is continuous for any $N\leqslant q<\infty$. From the Hölder inequality, (H_2) and Eq. (10), we have

$$\int_{\mathbb{R}^N}|u|\,dx\leqslant \left(\int_{\mathbb{R}^N}\frac{1}{(V(x))^{1/(N-1)}}dx\right)^{1-\frac{1}{N}}\left(\int_{\mathbb{R}^N}V(x)|u|^N dx\right)^{\frac{1}{N}}\leqslant \left(\int_{\mathbb{R}^N}\frac{1}{(V(x))^{1/(N-1)}}dx\right)^{1-\frac{1}{N}}\cdot \left(\int_{\mathbb{R}^N}(|\nabla u|^N+V(x)|u|^N)dx\right)^{\frac{1}{N}}\leqslant \left(\int_{\mathbb{R}^N}\frac{1}{(V(x))^{1/(N-1)}}dx\right)^{1-\frac{1}{N}}C_{N,p}\|u\|_{E,\alpha}.$$

Here and throughout this paper $C_{N,p}=(\lambda_{N,p}/(\lambda_{N,p}-\alpha))^{1/N}$ and $\lambda_{N,p}$ is defined by Eq. (10). Similarly, for any $1<\gamma<N$, we can also obtain

$$\int_{\mathbb{R}^N}|u|^\gamma dx\leqslant\int_{\mathbb{R}^N}(|u|+|u|^N)dx=\int_{\mathbb{R}^N}|u|\,dx+\int_{\mathbb{R}^N}\frac{1}{V(x)}V(x)|u|^N dx\leqslant \left(\int_{\mathbb{R}^N}\frac{1}{(V(x))^{1/(N-1)}}dx\right)^{1-\frac{1}{N}}C_{N,p}\|u\|_{E,\alpha}+\frac{1}{V_0}\int_{\mathbb{R}^N}V(x)|u|^N dx\leqslant \left(\int_{\mathbb{R}^N}\frac{1}{(V(x))^{1/(N-1)}}dx\right)^{1-\frac{1}{N}}C_{N,p}\|u\|_{E,\alpha}+\frac{1}{V_0}\int_{\mathbb{R}^N}(|\nabla u|^N+V(x)|u|^N)dx\leqslant \left(\int_{\mathbb{R}^N}\frac{1}{(V(x))^{1/(N-1)}}dx\right)^{1-\frac{1}{N}}C_{N,p}\|u\|_{E,\alpha}+\frac{1}{V_0}C_{N,p}^N\|u\|_{E,\alpha}^N,$$

where V_0 is given by (H_1). Consequently, we conclude that the embedding $E\hookrightarrow L^q(\mathbb{R}^N)$ is continuous.

Secondly, we will show that the embedding $E\hookrightarrow L^q(\mathbb{R}^N)$ is also compact. Choose a sequence of functions $\{u_n\}\subset E$ with $\|u_n\|_{E,\alpha}\leqslant C$ for any n, up to a subsequence of $\{u_n\}$ which we denote again by $\{u_n\}$, we need to show that there exists $u\in E$ such that $u_n\to u$ strongly in $L^q(\mathbb{R}^N)$ for any $q\geqslant 1$. Without loss of generality, we suppose

$$\left.\begin{array}{l} u_n\rightharpoonup u \text{ weakly in } E\\ u_n\to u \text{ strongly in } L^q_{\mathrm{loc}}(\mathbb{R}^N),\ \forall q\geqslant 1\\ u_n\to u \text{ almost everywhere in } \mathbb{R}^N\end{array}\right\}\quad(15)$$

Applying (H_2), for any $\epsilon>0$, there exists $R>0$ such that

$$\left(\int_{|x|>R}\frac{1}{(V(x))^{1/(N-1)}}dx\right)^{1-\frac{1}{N}}<\epsilon.$$

Therefore, by the Hölder inequality and Eq. (10), we get

$$\int_{|x|>R}|u_n-u|\,dx\leqslant \left(\int_{|x|>R}\frac{1}{(V(x))^{1/(N-1)}}dx\right)^{1-\frac{1}{N}}\cdot \left(\int_{\mathbb{R}^N}V(x)|u_n-u|^N dx\right)^{\frac{1}{N}}\leqslant \epsilon C_{N,p}\|u_n-u\|_{E,\alpha}\leqslant C\epsilon \quad(16)$$

Here and in the following, we often denote various constants by the same C. On the other hand, by Eq. (15), we have that $u_n\to u$ strongly in $L^1(\mathbb{B}_R)$, where $\mathbb{B}_R$ is the ball with radius R. Combining this with Eq. (16), we obtain

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|\,dx\leqslant C\epsilon.$$

Note that ϵ is arbitrary, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|\,dx=0.$$

For any $q>1$, in view of the Hölder inequality and the continuous of the embedding $E\hookrightarrow L^s(\mathbb{R}^N)$ $(s\geqslant 1)$, one has

$$\int_{\mathbb{R}^N}|u_n-u|^q dx=\int_{\mathbb{R}^N}|u_n-u|^{\frac{1}{2}}|u_n-u|^{q-\frac{1}{2}}dx\leqslant$$

$$\left(\int_{\mathbb{R}^N}|u_n-u|\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|u_n-u|^{2q-1}dx\right)^{\frac{1}{2}}\leqslant C\left(\int_{\mathbb{R}^N}|u_n-u|\,dx\right)^{\frac{1}{2}}\to 0 \text{ as } n\to\infty.$$

This concludes the desired result.

2 Variational framework

In order to use the critical point theory, in this section our main focus is variational framework of the Eq. (6). Now, Let us define a functional from E to $\mathbb{R}$ associated to the Eq. (6) as

$$J_{\alpha,\epsilon}(u)=\frac{1}{N}\int_{\mathbb{R}^N}(|\nabla u|^N+V(x)|u|^N)\,dx-\frac{\alpha}{N}\left(\int_{\mathbb{R}^N}|u|^p dx\right)^{\frac{N}{p}}-\int_{\mathbb{R}^N}\frac{F(x,u)}{|x|^\beta}dx-\epsilon\int_{\mathbb{R}^N}hu\,dx \tag{17}$$

where $F(x,s)=\int_0^s f(x,t)dt$ is a primitive of $f(x,s)$. It is easy to see that $J_{\alpha,\epsilon}(0)=0$. Moreover, by Proposition 1 in Ref. [14] and standard arguments in Ref. [23], we deduce that $J_{\alpha,\epsilon}(u)\in C^1(E,\mathbb{R})$. A simple calculation gives

$$\langle J'_{\alpha\epsilon}(u),\varphi\rangle=\int_{\mathbb{R}^N}(|\nabla u|^{N-2}\nabla u\nabla\varphi+V(x)|u|^{N-2}u\varphi)\,dx-\alpha\|u\|^{N-p}_{L^{\frac{N}{p}}(\mathbb{R})}\int_{\mathbb{R}^N}|u|^{p-2}u\varphi\,dx-\int_{\mathbb{R}^N}\frac{f(x,u)}{|x|^\beta}\varphi\,dx-\epsilon\int_{\mathbb{R}^N}h\varphi\,dx \tag{18}$$

for any $\varphi\in E$. Hence $u\in E$ is a critical point of the functional $J_{\alpha,\epsilon}(u)$ if and only if u is a positive weak solution of the equation (6). Therefore, to find the existence of positive weak solutions for the equation (6), we just focus on studying the existence of critical points of the functional $J_{\alpha,\epsilon}(u)$ on E. Now, we give several necessary lemmas on functional $J_{\alpha,\epsilon}(u)$, which are needed in the proof of our main result.

Lemma 2.1 Suppose that the conditions (H_1), (H_3)~(H_6) are satisfied. Then, we have

(Ⅰ$_1$) $J_{\alpha,\epsilon}(tu)\to-\infty$ as $t\to+\infty$ for every compactly supported function $u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}$;

(Ⅰ$_2$) there exists $\epsilon_1>0$ such that for any $\epsilon\in(0,\epsilon_1)$, we can find some r_ϵ, $\vartheta_\epsilon>0$ such that $J_{\alpha,\epsilon}(u)\geqslant\vartheta_\epsilon$ for any u with $\|u\|_{E,\alpha}=r_\epsilon$, where r_ϵ can be taken such that $r_\epsilon\to 0$ as $\epsilon\to 0$;

(Ⅰ$_3$) suppose $\epsilon>0$, $h\not\equiv 0$, there exists a real constant $\tau>0$ such that $\inf_{\|u\|_{E,\alpha}\leqslant t}J_{\alpha,\epsilon}(u)<0$ for any $t\in(0,\tau)$.

Proof According to (H_4) and (H_5), there exists a positive real constant R_0 such that for any $(x,s)\in\mathbb{R}^N\times[R_0,\infty)$, $F(x,s)>0$ and $\mu F(x,s)\leqslant s\frac{\partial F(x,s)}{\partial s}$, which shows

$$\frac{\partial \ln F(x,s)}{\partial s}\leqslant\frac{\mu}{s}.$$

Now we integrate the above inequality from R_0 to μ on both sides on s to get $F(x,s)\geqslant\frac{F(x,R_0)}{R_0^\mu}s^\mu$. Suppose u is supported in a bounded domain Ω. Therefore, for any $(x,s)\in\Omega\times[R_0,\infty)$, there exists a positive real constant b_1 such that $F(x,s)\geqslant b_1 s^\mu$. According to (H_4), also notice that $f(x,s)$ is a continuous function, for any $(x,s)\in\Omega\times[0,R_0]$, there exists a positive real constant b_2 such that $F(x,s)\geqslant -b_2$. Thus, for any $(x,s)\in\Omega\times[0,\infty)$, we know that $F(x,s)\geqslant b_1 s^\mu-b_2$, which implies

$$J_{\alpha,\epsilon}(tu)=\frac{1}{N}\int_\Omega(|\nabla tu|^N+V(x)|tu|^N)\,dx-\frac{\alpha}{N}\left(\int_\Omega|tu|^p dx\right)^{\frac{N}{p}}-\int_\Omega\frac{F(x,tu)}{|x|^\beta}dx-\epsilon\int_\Omega htu\,dx\leqslant\frac{t^N}{N}\|u\|^N_{E,\alpha}-b_1t^\mu\int_\Omega\frac{|\mu|^N}{|x|^\beta}dx+b_2\int_\Omega\frac{1}{|x|^\beta}dx-\epsilon t\int_\Omega hu\,dx.$$

Observing that $\mu>N\geqslant 2$ and $b_1>0$, we have $J_{\alpha,\epsilon}(tu)\to-\infty$ as $t\to+\infty$. From the aforementioned discussion, it is obvious that the conclusion is also suitable for $J_\alpha(u)$. Hence, property (Ⅰ$_1$) holds.

Now, applying (H_6), we deduce that there are two positive real constants τ and δ such that for $|s|\leqslant\delta$, $x\in\mathbb{R}^N$, then

$$|F(x,s)|\leqslant\frac{(\lambda_\beta-\tau)|s|^N}{N}.$$

On the other hand, owing to (H_3), when

$|s|\geqslant\delta$, we have

$$F(x,s)\leqslant\int_0^{|s|}(a_1 t^{N-1}+a_2\zeta(N,\alpha_0 t^{\frac{N}{N-1}}))\,\mathrm{d}t\leqslant \frac{a_1|s|^N}{N}+a_2\zeta(N,\alpha_0 s^{\frac{N}{N-1}})|s|\leqslant c_{\delta,N}(\zeta(N,\alpha_0 s^{\frac{N}{N-1}}))|s|^{N+1},$$

where $c_{\delta,N}\geqslant\frac{a_1}{\delta N\zeta(N,\alpha_0 s^{N/(N-1)})}+\frac{a_2}{\delta^N}$. Then for any $(x,s)\in\mathbb{R}^N\times\mathbb{R}$ and by applying the above mentioned two inequalities, one can write that

$$F(x,s)\leqslant\frac{(\lambda_\beta-\tau)|s|^N}{N}+C(\zeta(N,\alpha_0 s^{\frac{N}{N-1}}))|s|^{N+1}\tag{19}$$

Next we will show that

$$\int_{\mathbb{R}^N}\frac{|u|^{N+1}\zeta(N,\alpha_0|u|^{\frac{N}{N-1}})}{|x|^\beta}\mathrm{d}x\leqslant C\|u\|_{E,\alpha}^{N+1}\tag{20}$$

Therefore, we suppose u^* is the Schwarz rearrangement of $|u|$. It follows from Theorem 2.1 in Ref. [11] (Hardy-Littlewood inequality) that

$$\int_{\mathbb{R}^N}\frac{|u|^{N+1}\zeta(N,\alpha_0|u|^{\frac{N}{N-1}})}{|x|^\beta}\mathrm{d}x\leqslant \int_{\mathbb{R}^N}\frac{|u^*|^{N+1}\zeta(N,\alpha_0|u^*|^{\frac{N}{N-1}})}{|x|^\beta}\mathrm{d}x\tag{21}$$

Let a positive real number

$$\gamma>2\left(\frac{N}{\omega_{N-1}}\right)^{1/N}\|u\|_{L^N(\mathbb{R}^N)}\tag{22}$$

For any positive real constant c, c', d, d' with $c>1$, $1<d<N/\beta$, $1/c+1/c'=1$, $1/d+1/d'=1$, by the Hölder inequality, we have

$$\int_{|x|\leqslant\gamma}\frac{|u^*|^{N+1}\zeta(N,\alpha_0|u^*|^{\frac{N}{N-1}})}{|x|^\beta}\mathrm{d}x\leqslant \int_{|x|\leqslant\gamma}\frac{|u^*|^{N+1}e^{\alpha_0|u^*|^{\frac{N}{N-1}}}}{|x|^\beta}\mathrm{d}x\leqslant$$
$$\left(\int_{|x|\leqslant\gamma}\frac{e^{c\alpha_0|u^*|^{\frac{N}{N-1}}}}{|x|^\beta}\mathrm{d}x\right)^{\frac{1}{c}}\left(\int_{|x|\leqslant\gamma}\frac{1}{|x|^{\beta d}}\mathrm{d}x\right)^{\frac{1}{c'd}}\cdot \left(\int_{|x|\leqslant\gamma}|u^*|^{(N+1)c'd'}\mathrm{d}x\right)^{\frac{1}{c'd'}}\leqslant$$
$$C\left(\int_{\mathbb{R}^N}\frac{\zeta(N,c\alpha_0|u^*|^{\frac{N}{N-1}})}{|x|^\beta}\mathrm{d}x\right)^{\frac{1}{c}}\cdot \left(\int_{\mathbb{R}^N}|u^*|^{(N+1)c'd'}\mathrm{d}x\right)^{\frac{1}{c'd'}}.$$

Adding Eq. (11) and the continuous embedding $E\hookrightarrow L^q(\mathbb{R}^N)$ $(N\leqslant q<\infty)$ in Lemma 1.2 and choosing $\|u\|_{E,\alpha}$ small enough such that $c\alpha_0\|u\|_{E,\alpha}^{\frac{N}{N-1}}\leqslant\alpha_N$, we obtain

$$\int_{|x|\leqslant\gamma}\frac{|u^*|^{N+1}\zeta(N,\alpha_0|u^*|^{\frac{N}{N-1}})}{|x|^\beta}\mathrm{d}x\leqslant C\|u\|_{E,\alpha}^{N+1}\tag{23}$$

where C depends only on N, γ, β. By Eq. (22), the radial lemma and the continuous embedding $E\hookrightarrow L^{N+1}(\mathbb{R}^N)$ (see Lemma 1.2), we can deduce that for some positive real constant C

$$\int_{|x|\geqslant\gamma}\frac{|u^*|^{N+1}\zeta(N,\alpha_0|u^*|^{\frac{N}{N-1}})}{|x|^\beta}\mathrm{d}x\leqslant \frac{\zeta(N,\alpha_0|u^*(\gamma)|^{\frac{N}{N-1}})}{|\gamma|^\beta}\int_{|x|\geqslant\gamma}|u^*|^{N+1}\mathrm{d}x\leqslant$$
$$\frac{\zeta\left(N,\alpha_0\left(\frac{1}{2}\right)^{\frac{N}{N-1}}\right)}{|\gamma|^\beta}\|u^*\|_{L^{N+1}(\mathbb{R}^N)}^{N+1}\leqslant C\|u\|_{E,\alpha}^{N+1}\tag{24}$$

It follows from Eqs. (21), (23) and (24) that Eq. (20) holds. In fact, Eq. (12) implies for all $u\in E, u\neq0$, $0\leqslant\beta<N$

$$\lambda_\beta\int_{\mathbb{R}^N}\frac{|u|^N}{|x|^\beta}\mathrm{d}x\leqslant\|u\|_{E,\alpha}\tag{25}$$

Also by (H_1), we have

$$\|u\|_{L^N(\mathbb{R}^N)}=\left(\int_{\mathbb{R}^N}\frac{1}{V(x)}V(x)|u|^N\mathrm{d}x\right)^{\frac{1}{N}}\leqslant \frac{1}{V_0^{1/N}}\left(\int_{\mathbb{R}^N}V(x)|u|^N\mathrm{d}x\right)^{\frac{1}{N}}\leqslant$$
$$\frac{1}{V_0^{1/N}}\left(\int_{\mathbb{R}^N}(|\nabla u|^N+V(x)|u|^N)\mathrm{d}x\right)^{\frac{1}{N}}\leqslant \frac{C_{N,p}}{V_0^{1/N}}\|u\|_{E,\alpha}.$$

This together with Eqs. (19), (20), (25) and the Hölder inequality leads to

$$J_{\alpha,\epsilon}(u)=\frac{1}{N}\|u\|_{E,\alpha}^N-\int_{\mathbb{R}^N}\frac{F(x,u)}{|x|^\beta}\mathrm{d}x-\epsilon\int_{\mathbb{R}^N}hu\,\mathrm{d}x\geqslant$$
$$\frac{1}{N}\|u\|_{E,\alpha}^N-\frac{\lambda_\beta-\tau}{N}\int_{\mathbb{R}^N}\frac{|u|^N}{|x|^\beta}\mathrm{d}x- C\|u\|_{E,\alpha}^{N+1}-\epsilon\|h\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)}\|u\|_{L^N(\mathbb{R}^N)}\geqslant$$

$$\frac{\tau}{N\lambda_\beta}\|u\|_{E,\alpha}^{N}-C\|u\|_{E,\alpha}^{N+1}-\frac{\epsilon C_{N,p}}{V_0^{1/N}}\|u\|_{E,\alpha}\|h\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)}=$$

$$\|u\|_{E,\alpha}\left(\frac{\tau}{N\lambda_\beta}\|u\|_{E,\alpha}^{N-1}-C\|u\|_{E,\alpha}^{N}-\frac{\epsilon C_{N,p}}{V_0^{1/N}}\|h\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)}\right) \tag{26}$$

Notice that $\tau>0$, then there exists a small enough real constant $r>0$ such that

$$\frac{\tau}{N\lambda_\beta}r^{N-1}-Cr^N\geqslant\frac{\tau}{2N\lambda_\beta}r^{N-1} \tag{27}$$

Also notice that $h\not\equiv0$, for small enough $\epsilon>0$, first of all we choose r_ϵ such that

$$\frac{\tau}{2N\lambda_\beta}r_\epsilon^{\ N-1}=\frac{2\epsilon C_{N,p}}{V_0^{1/N}}\|h\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)},$$

then we let

$$\vartheta_\epsilon=\frac{\epsilon C_{N,p}}{V_0^{1/N}}\|u\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)}.$$

From Eq. (26) and the above two equalities, we derive $J_{\alpha,\epsilon}(u)\geqslant\vartheta_\epsilon$ for any u with $\|u\|_{E,\alpha}=r_\epsilon$ and $r_\epsilon\to0$ as $\epsilon\to0$. Thus, property (I_2) is true.

By the standard method bases on variation, we may suppose that υ is a weak solution of

$$-\mathrm{div}(|\nabla\upsilon|^{N-2}\nabla\upsilon)+V(x)|\upsilon|^{N-2}\upsilon-\alpha\|\upsilon\|_{L^p(\mathbb{R}^N)}^{N-p}|\upsilon|^{p-2}\upsilon=\epsilon h\ \text{ in }\mathbb{R}^N.$$

Let us observe that $h\not\equiv0$, then $\epsilon\int_{\mathbb{R}^N}h\upsilon\,\mathrm{d}x=\|\upsilon\|_{E,\alpha}^{N}>0$. For any $t>0$, we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}J_{\alpha,\epsilon}(t\upsilon)=t^{N-1}\|\upsilon\|_{E,\alpha}^{N}-\int_{\mathbb{R}^N}\frac{f(x,t\upsilon)}{|x|^\beta}\upsilon\,\mathrm{d}x-\epsilon\int_{\mathbb{R}^N}h\upsilon\,\mathrm{d}x.$$

This together with $f(x,0)=0$, we obtain $\frac{\mathrm{d}}{\mathrm{d}t}J_{\alpha,\epsilon}(t\upsilon)|_{t=0}<0$. Using continuity yields there is $\tau>0$ such that $\frac{\mathrm{d}}{\mathrm{d}t}J_{\alpha,\epsilon}(t\upsilon)<0$ for any $t\in(0,\tau)$. Since $J_{\alpha,\epsilon}=0$, we have $J_{\alpha,\epsilon}(t\upsilon)<0$ for any $t\in(0,\tau)$, which ends the proof of property (I_3).

In order to obtain the min-max level of the functional $J_{\alpha,\epsilon}(u)$, we recall the function sequence of Moser

$$\widetilde{M}_n(x,r)=\frac{1}{\omega_{N-1}^{1/N}}\begin{cases}(\log n)^{1-1/N}, & \text{if } |x|\leqslant\frac{r}{n},\\ \dfrac{\log\frac{r}{|x|}}{(\log n)^{1/N}}, & \text{if } \frac{r}{n}\leqslant|x|\leqslant r,\\ 0, & \text{if } |x|>r,\end{cases}$$

where ω_{N-1} is given by Eq. (1). Let $M_n(x,r)=\widetilde{M}_n(x,r)/\|\widetilde{M}_n\|_{E,\alpha}$. Obviously, $\|M_n\|_{E,\alpha}=1$ and $M_n\in E$ with its support in $B_R(0)$.

Lemma 2.2 Suppose that $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ and the condition (H_1) holds. Then we have

$$\|\widetilde{M}_n(x,r)\|_{E,\alpha}^{N}\leqslant 1+\frac{m(r)}{\log n}\left(\frac{(N-1)!}{N^N}r^N+o_n(1)\right)-\alpha\|\widetilde{M}_n(x,r)\|_{L^p(\mathbb{R}^N)}^{N},$$

where $m(r)=\max_{|x|\leqslant r}V(x)$ and $o_n(1)\to0$ as $n\to\infty$.

Proof By a simple calculation, we have

$$\int_{\mathbb{R}^N}(|\nabla\widetilde{M}_n(x,r)|^N)\,\mathrm{d}x=\frac{1}{\omega_{N-1}\log n}\int_{\mathbb{R}^N}\frac{1}{|x|^N}\mathrm{d}x=1$$

and

$$\int_{\mathbb{R}^N}(|\widetilde{M}_n(x,r)|^N)\,\mathrm{d}x=\int_{|x|\leqslant\frac{r}{n}}\frac{(\log n)^{N-1}}{\omega_{N-1}}\mathrm{d}x+\int_{\frac{r}{n}\leqslant|x|\leqslant r}\frac{\left(\log\frac{r}{|x|}\right)^N}{\omega_{N-1}\log n}\mathrm{d}x=$$

$$\left(\frac{r}{n}\right)^N\frac{(\log n)^{N-1}}{N}+\frac{1}{\log n}\left(\frac{(N-1)!}{N^N}r^N+o_n(1)\right)=\frac{1}{\log n}\left(\frac{(N-1)!}{N^N}r^N+o_n(1)\right).$$

Here we have used the integration by parts. Consequently, we get

$$\|\widetilde{M}_n(x,r)\|_{E,\alpha}^{N}=\int_{\mathbb{R}^N}(|\nabla\widetilde{M}_n(x,r)|^N+V(x)|\widetilde{M}_n(x,r)|^N)\,\mathrm{d}x-\alpha\|\widetilde{M}_n(x,r)\|_{L^p(\mathbb{R}^N)}^{N}\leqslant$$

$$1+\frac{m(r)}{\log n}\left(\frac{(N-1)!}{N^N}r^N+o_n(1)\right)-\alpha\|\widetilde{M}_n(x,r)\|_{L^p(\mathbb{R}^N)}^{N}.$$

Lemma 2.3 Suppose that the conditions (H_1), (H$_3$), (H$_4$), (H$_5$) and (H$_7$) are satisfied. Then there exists a positive integer n such that

$$\max_{t\geqslant 0} J_\alpha(tM_n) < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Also for the aforementioned n, there exist two real constants ϵ^* and δ^* such that, for any $\epsilon\in[0,\epsilon^*)$, we have

$$\max_{t\geqslant 0} J_{\alpha,\epsilon}(tM_n) < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} - \delta^*.$$

Proof Since the functionals $J_\alpha(u)$ and $J_{\alpha,\epsilon}(u)$ are smaller than those in Ref. [15], the inequalities in the lemma are trivial.

Next, we will prove an analog of (Ref. [15], Lemma 2.4) as follows:

Lemma 2.4 Suppose that $\{u_n\}\subset E$ and the conditions (H$_1$)~(H$_5$) are satisfied. if $\{u_n\}$ is a Palais-Smale sequence of $J_{\alpha,\epsilon}$, that is

$$\lim_{n\to\infty} J_{\alpha,\epsilon}(u_n) = c \text{ in } E \tag{28}$$

and

$$\lim_{n\to\infty} J'\alpha_\epsilon(u_n) = 0 \text{ in } E^* \tag{29}$$

Then, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u\in E$ such that $u_n\rightharpoonup u$ weakly in E, $u_n\to u$ strongly in $L^q(\mathbb{R}^N)$ for any $q\geqslant 1$, and

$$\left.\begin{aligned} &\frac{f(x,u_n)}{|x|^\beta}\to\frac{f(x,u)}{|x|^\beta} \text{ strongly in } L^1(\mathbb{R}^N),\\ &\frac{F(x,u_n)}{|x|^\beta}\to\frac{F(x,u)}{|x|^\beta} \text{ strongly in } L^1(\mathbb{R}^N),\\ &\nabla u_n(x)\to\nabla u(x) \text{ almost everywhere in } \mathbb{R}^N.\end{aligned}\right\}$$

In addition, u is a weak solution of Eq. (6).

Proof Observing (H$_4$) and Eq. (8), we obtain for some $\mu>N$

$$0\leqslant \mu F(x,u_n)\leqslant u_n f(x,u_n) \tag{30}$$

Next, we suppose that $\{u_n\}$ is a Palais-Smale sequence of $J_{\alpha,\epsilon}$. Then, from Eqs. (17) and (28) it follows that

$$J_{\alpha,\epsilon}(u_n) = \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla u_n|^N + V(x)|u_n|^N)\,dx - \frac{\alpha}{N}\left(\int_{\mathbb{R}^N}|u_n|^p dx\right)^{\frac{N}{p}} - \int_{\mathbb{R}^N}\frac{F(x,u_n)}{|x|^\beta}dx - \epsilon\int_{\mathbb{R}^N} h u_n dx = \frac{1}{N}\|u_n\|_{E,\alpha}^N - \int_{\mathbb{R}^N}\frac{F(x,u_n)}{|x|^\beta}dx - \epsilon\int_{\mathbb{R}^N} h u_n dx \to c \text{ as } n\to\infty.$$

Multiplying both sides of the last equation by μ, we have

$$\frac{\mu}{N}\|u_n\|_{E,\alpha}^N - \int_{\mathbb{R}^N}\frac{\mu F(x,u_n)}{|x|^\beta}dx - \mu\epsilon\int_{\mathbb{R}^N} h u_n dx = \mu c + o_n(1) \tag{31}$$

where $o_n(1)\to 0$ as $n\to\infty$. Also from Eqs. (18) and (19), we get for any $\varphi\in E$

$$\langle J'\alpha_\epsilon(u_n),\varphi\rangle = \int_{\mathbb{R}^N}(|\nabla u_n|^{N-2}\nabla u_n\nabla\varphi + V(x)|u_n|^{N-2}u_n\varphi)dx - \alpha\|u_n\|_{L^p(\mathbb{R}^N)}^{N-p}\int_{\mathbb{R}^N}|u|^{p-2}u_n\varphi dx - \int_{\mathbb{R}^N}\frac{f(x,u_n)}{|x|^\beta}\varphi dx - \epsilon\int_{\mathbb{R}^N}h\varphi dx\to 0 \text{ as } n\to\infty \tag{32}$$

this together with Eq. (18) gives

$$|\langle J'\alpha_\epsilon(u_n),\varphi\rangle|\leqslant a_n\|\varphi\|_{E,\alpha},\ \forall\varphi\in E \tag{33}$$

where $a_n\to 0$ as $n\to\infty$. Choosing $\varphi=u_n$ in the last inequality, we can conclude that

$$-\left(\int_{\mathbb{R}^N}(|\nabla u_n|^N + V(x)|u_n|^N)dx - \alpha\|u_n\|_{L^p(\mathbb{R}^N)}^{N-p}\int_{\mathbb{R}^N}|u_n|^p dx\right) + \int_{\mathbb{R}^N}\frac{u_n f(x,u_n)}{|x|^\beta}dx + \epsilon\int_{\mathbb{R}^N}h u_n dx \leqslant a_n\|u_n\|_{E,\alpha} \tag{34}$$

We have by combining Eqs. (30), (31) and (34),

$$\left(\frac{\mu}{N}-1\right)\|u_n\|_{E,\alpha}^N \leqslant \left(\frac{\mu}{N}-1\right)\|u_n\|_{E,\alpha}^N + \int_{\mathbb{R}^N}\frac{u_n f(x,u_n)-\mu F(x,u_n)}{|x|^\beta}dx \leqslant \mu|c| + a_n\|u_n\|_{E,\alpha} + \epsilon(\mu-1)\|h\|_{L^{p'}(\mathbb{R}^N)}\|u_n\|_{L^p(\mathbb{R})^N},$$

where $\mu>N$, $p\geqslant N$, $N\geqslant 2$ and $1/p+1/p'=1$. Hence $\|u_n\|_{E,\alpha}$ is bounded, which together with Eq. (31) yields

$$\int_{\mathbb{R}^N}\frac{F(x,u_n)}{|x|^\beta}dx\leqslant C \tag{35}$$

for some constant C. Thus Eq. (34) yields

$$\int_{\mathbb{R}^N}\frac{u_n f(x,u_n)}{|x|^\beta}dx\leqslant C \tag{36}$$

By Lemma 1.2, up to a subsequence, $u_n \to u$ strongly in $L^q(\mathbb{R}^N)$ for some $u \in E$ and any $q \geqslant 1$, which means that $u_n \to u$ almost everywhere in $\mathbb{R}^N$. Since u, $f(x, u)/|x|^\beta \in L^1(\mathbb{R}^N)$, it follows that

$$\lim_{\rho \to +\infty} \int_{|u|>\rho} \frac{f(x,u)}{|x|^\beta} \mathrm{d}x = 0.$$

Let C be the constant given in Eq. (36). Again by (H_4) and Eq. (8), we have

$$f(x, u) \geqslant 0,\ uf(x,u) = |u| f(x, |u|).$$

Then, for any given $\epsilon > 0$ there exists $K > C/\epsilon$ such that

$$\int_{|u| \geqslant K} \frac{f(x,u)}{|x|^\beta} \mathrm{d}x \leqslant \frac{1}{K} \int_{|u| \geqslant K} \frac{uf(x,u)}{|x|^\beta} \mathrm{d}x \leqslant \frac{C}{K} < \epsilon \tag{37}$$

Similarly we have

$$\int_{|u_n| \geqslant K} \frac{f(x,u_n)}{|x|^\beta} \mathrm{d}x \leqslant \frac{1}{K} \int_{|u_n| \geqslant K} \frac{u_n f(x,u_n)}{|x|^\beta} \mathrm{d}x \leqslant \frac{C}{K} < \epsilon \tag{38}$$

For any $x \in \mathbb{R}^N$ and $|u_n(x)| < K$, according to (H_3), there is a constant C_0 depending only on K such that $|f(x, u_n)| \leqslant C_0 |u_n|^{N-1}$. Since $|u_n|^{N-1}/|x|^\beta \to |u|^{N-1}/|x|^\beta$ strongly in $L^1(\mathbb{R}^N)$ and $u_n \to u$ almost everywhere in $\mathbb{R}^N$, it follows from the generalized Lebesgue's dominated convergence theorem that

$$\lim_{n \to \infty} \int_{|u_n|<K} \frac{f(x,u_n)}{|x|^\beta} \mathrm{d}x = \int_{|u|<K} \frac{f(x,u)}{|x|^\beta} \mathrm{d}x,$$

that is, for any given $\epsilon > 0$ one can take some positive integer n_0 such that when $n > n_0$

$$\left| \int_{|u_n|<K} \frac{f(x,u_n)}{|x|^\beta} \mathrm{d}x - \int_{|u_n|<K} \frac{f(x,u)}{|x|^\beta} \mathrm{d}x \right| < \epsilon.$$

This together with Eqs. (37), (38) and $f(x, u) \geqslant 0$ leads to

$$\left| \int_{\mathbb{R}^N} \frac{f(x,u_n)}{|x|^\beta} \mathrm{d}x - \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^\beta} \mathrm{d}x \right| \leqslant \left| \int_{|u_n| \geqslant K} \frac{f(x,u_n)}{|x|^\beta} \mathrm{d}x - \int_{|u_n| \geqslant K} \frac{f(x,u)}{|x|^\beta} \mathrm{d}x \right| + \left| \int_{|u_n|<K} \frac{f(x,u_n)}{|x|^\beta} \mathrm{d}x - \int_{|u_n|<K} \frac{f(x,u)}{|x|^\beta} \mathrm{d}x \right| \leqslant \left| \int_{|u_n| \geqslant K} \frac{f(x,u_n)}{|x|^\beta} \mathrm{d}x + \int_{|u_n| \geqslant K} \frac{f(x,u)}{|x|^\beta} \mathrm{d}x + \right. \left| \int_{|u_n|<K} \frac{f(x,u_n)}{|x|^\beta} \mathrm{d}x - \int_{|u_n|<K} \frac{f(x,u)}{|x|^\beta} \mathrm{d}x \right| < 3\epsilon,$$

which yields

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)}{|x|^\beta} \mathrm{d}x = \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^\beta} \mathrm{d}x.$$

Noting also that $f(x,u_n) \geqslant 0$, we obtain

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|f(x,u_n) - f(x,u)|}{|x|^\beta} \mathrm{d}x = 0 \tag{39}$$

Thanks to (H_3) and (H_5), there exist two positive constants c_1, c_2 such that $F(x, u_n) \leqslant c_1 |u_n|^N + c_2 f(x, u_n)$. Thus, by Eq. (39), Lemma 1.2 and the generalized Lebesgue's dominated convergence theorem, we get

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|F(x,u_n) - F(x,u)|}{|x|^\beta} \mathrm{d}x = 0.$$

Next by applying the proof of (Ref. [11] Eq. (4.26)), we derive

$$\nabla u_n(x) \to \nabla u(x) \text{ almost everywhere in } \mathbb{R}^N$$

and

$$|\nabla u_n|^{N-2} \nabla u_n \rightharpoonup |\nabla u|^{N-2} \nabla u \text{ weakly in } (L^{\frac{N}{N-1}}(\mathbb{R}^N))^N.$$

Letting $n \to \infty$ in Eq. (32), we get

$$\int_{\mathbb{R}^N} (|\nabla u|^{N-2} \nabla u \nabla \varphi + V(x) |u|^{N-2} u\varphi) \mathrm{d}x - \alpha \| u \|_{L^p(\mathbb{R}^N)}^{N-p} \int_{\mathbb{R}^N} |u|^{p-2} u\varphi \mathrm{d}x - \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^\beta} \varphi \mathrm{d}x - \epsilon \int_{\mathbb{R}^N} h\varphi \mathrm{d}x = 0$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, which is dense in E. Thus u is a weak solution of Eq. (6).

3 Proof of Theorem 0.1

In this section, we are ready to prove Theorem 0.1 by applying the mountain-pass theorem without the Palais-Smale condition. It suffices to prove that the functional $J_{\alpha,\epsilon}$ has two distinct critical points in E.

Proof of Theorem 0.1 First of all, we will prove that the equation (6) has a mountain-pass type weak solution. Suppose that ϵ_1 is given by (I_1) of Lemma 2.1, and ϵ^*, δ^* are given by Lemma 2.3. According to (I_1) and (I_2) of Lemma 2.1, for $\epsilon \in (0, \epsilon_1)$, one can easily find that $J_{\alpha,\epsilon}(u) \in C^1(E, \mathbb{R})$, $J_{\alpha,\epsilon}(u) \geqslant \vartheta_\epsilon$ for

$\| u \|_{E,\alpha} = r_\epsilon$ and $\inf_{\| e \|_{E,\alpha} \leqslant t} J_{\alpha,\epsilon}(e) < 0$ for $t \in (0, \tau)$. Then, by the mountain-pass theorem without the Palais-Smale condition in Ref. [23], there exists a sequence $\{v_n\} \subset E$ such that

$$\lim_{n\to\infty} J_{\alpha,\epsilon}(v_n) = c_m \text{ in } E, \lim_{n\to\infty} J'_{\alpha,\epsilon}(v_n) = 0 \text{ in } E^* \quad (40)$$

here $c_m = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} J_{\alpha,\epsilon}(u) \geqslant \vartheta_\epsilon > 0$ is a minimax value of $J_{\alpha,\epsilon}$ with

$$\Gamma = \{\gamma \in C([0,1], E), \gamma(0) = 0, \gamma(1) = e\}.$$

Take $\epsilon_2 = \min\{\epsilon_1, \epsilon^*\}$. Then for any $\epsilon \in (0, \epsilon_2)$, by Lemmas 2.3 and 2.4, we obtain that

$$0 < c_m < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} - \delta^* \quad (41)$$

and that up to a subsequence, v_n converges weakly in E to a weak solution u_1 of Eq. (6).

Second, we will show that the Eq. (6) has a local minimum type weak solution u_2. Suppose that r_ϵ is given by (I_2) of Lemma 2.1. Then it follows from (I_2) of Lemma 2.1 that $J_{\alpha,\epsilon}(u) \geqslant \vartheta_\epsilon > 0$ for $\| u \|_{E,\alpha} = r_\epsilon$, and that

$$\lim_{n\to\infty} r_\epsilon = 0 \quad (42)$$

Hence there exists some $\epsilon_3 \in (0, \epsilon_2)$ such that for $\epsilon \in (0, \epsilon_3)$

$$r_\epsilon < \left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{\frac{N}{N-1}} \quad (43)$$

From (H_3) and (H_4), we get

$$F(x,u) \leqslant a_1 |u|^N + a_2 |u| \zeta\left(N, \alpha_0 \| u \|_{E,\alpha}^{\frac{N}{N-1}} \left(\frac{|u|}{\| u \|_{E,\alpha}}\right)^{\frac{N}{N-1}}\right) \quad (44)$$

If $\| u \|_{E,\alpha} \leqslant r_\epsilon$, we have

$$\alpha_0 \| u \|_{E,\alpha}^{\frac{N}{N-1}} < \left(1 - \frac{\beta}{N}\right)\alpha_N.$$

Then, if $\| u \|_{E,\alpha} \leqslant r_\epsilon$, by the last inequality, Eq. (11) and Ref. [15], Lemma 1.1, we get that $F(x, u)/|x|^\beta$ is bounded in $L^p(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Therefore $J_{\alpha,\epsilon}$ has lower bound on $B_{r_\epsilon} = \{u \in E: \| u \|_{E,\alpha} \leqslant r_\epsilon\}$. It is obvious that $\overline{B}_{r_\epsilon} \subset E$ is convex and $J_{\alpha,\epsilon} \in C^1(E, \mathbb{R})$ has lower bound on $\overline{B}_{r_\epsilon}$. It follows from Ekeland's variational principle[24] that there is a sequence $\{u_n\} \subset \overline{B}_{r_\epsilon}$ such that

$$\lim_{n\to\infty} J_{\alpha,\epsilon}(u_n) = c_\epsilon := \inf_{\| u \|_{E,\alpha} \leqslant r_\epsilon} J_{\alpha,\epsilon}(u) \text{ in } E,$$

$$\lim_{n\to\infty} J'_{\alpha,\epsilon}(u_n) = 0 \text{ in } E^* \quad (45)$$

Then, by combining (I_3) of Lemma 2.1, Eq. (44), Lemma 1.4 and the Hölder inequality, we get that $c_\epsilon < 0$,

$$\lim_{\epsilon\to 0} \sup_{\| u \|_{E,\alpha} \leqslant r_\epsilon} \int_{\mathbb{R}^N} \frac{F(x, u)}{|x|^\beta} dx = 0,$$

$$\lim_{\epsilon\to 0} \sup_{\| u \|_{E,\alpha} \leqslant r_\epsilon} \int_{\mathbb{R}^N} hu\, dx = 0,$$

which means that $\lim_{\epsilon\to 0} c_\epsilon = 0$. Suppose $u_n \rightharpoonup u_2$ weakly in E. Let us now substitute $\varphi = u_n - u_2$ into Eq. (33)

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{N-2} \nabla u_n \nabla(u_n - u_2) + V(x) |u_n|^{N-2} u_n (u_n - u_2)) dx - \alpha \| u_n \|_{L^p(\mathbb{R}^N)}^{N-p} \int_{\mathbb{R}^N} |u_n|^{p-2} u_n (u_n - u_2) dx - \int_{\mathbb{R}^N} \frac{f(x, u_n)}{|x|^\beta}(u_n - u_2) dx - \epsilon \int_{\mathbb{R}^N} h(u_n - u_2) dx \to 0 \text{ as } n \to \infty.$$

This fact, together with Eqs. (43), (11), Lemma 1.2 and the Hölder inequality leads to

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x, u_n)}{|x|^\beta}(u_n - u_2) dx = 0,$$

$$\lim_{n\to\infty} \epsilon \int_{\mathbb{R}^N} h(u_n - u_2) dx = 0.$$

So that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{N-2} \nabla u_n \nabla(u_n - u_2) + V(x) |u_n|^{N-2} u_n (u_n - u_2)) dx - \alpha \| u_n \|_{L^p(\mathbb{R}^N)}^{N-p} \int_{\mathbb{R}^N} |u_n|^{p-2} u_n (u_n - u_2) dx \to 0 \text{ as } n \to \infty \quad (46)$$

It follows from $u_n \rightharpoonup u_2$ weakly in E that

$$\int_{\mathbb{R}^N} (|\nabla u_2|^{N-2} \nabla u_2 \nabla(u_n - u_2) + V(x) |u_2|^{N-2} u_2 (u_n - u_2)) dx - \alpha \| u_2 \|_{L^p(\mathbb{R}^N)}^{N-p} \int_{\mathbb{R}^N} |u_2|^{p-2} u_2 (u_n - u_2) dx \to 0 \text{ as } n \to \infty \quad (47)$$

Subtracting Eq. (47) from Eq. (46), applying (I) in Chapter 10 of Ref. [25], we can deduce that $\| u_n - u_2 \|_{E,\alpha}^N \to 0$ as $n \to \infty$. Then $u_n \to u_2$ strongly in E. Also notice that $J_{\alpha,\epsilon}(u) \in C^1(E, \mathbb{R})$, one can easily find that $J_{\alpha,\epsilon}(u_2) = c_\epsilon$ and $J'_{\alpha,\epsilon}(u_2) = 0$. Therefore, by Lemma 2.4, it follows that u_2 is a weak solution of Eq. (6).

Finally, we will verify that two weak solutions u_1 and u_2 are distinct. Suppose by contradiction that $u_1 \equiv u_2$. It follows from Lemma 1.5 and the Hölder inequality that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h v_n \,\mathrm{d}x = \int_{\mathbb{R}^N} h u_2 \,\mathrm{d}x \tag{48}$$

By Lemma 2.4, we also have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \frac{F(x, v_n)}{|x|^\beta}\mathrm{d}x = \int_{\mathbb{R}^N} \frac{F(x, u_2)}{|x|^\beta}\mathrm{d}x \tag{49}$$

Substituting Eqs. (48) and (49) into Eq. (40), we have

$$\lim_{n\to\infty}\frac{1}{N}\|v_n\|_{E,\alpha}^N = c_m + \int_{\mathbb{R}^N} \frac{F(x, u_2)}{|x|^\beta}\mathrm{d}x + \epsilon\int_{\mathbb{R}^N} h u_2 \,\mathrm{d}x \tag{50}$$

Similarly, we can also obtain

$$\lim_{n\to\infty}\frac{1}{N}\|u_n\|_{E,\alpha}^N = c_\epsilon + \int_{\mathbb{R}^N} \frac{F(x, u_2)}{|x|^\beta}\mathrm{d}x + \epsilon\int_{\mathbb{R}^N} h u_2 \,\mathrm{d}x.$$

This together with Eq. (50) and $\lim_{n\to\infty}\|u_n - u_2\|_{E,\alpha}^N = 0$ gives

$$\lim_{n\to\infty}(\|v_n\|_{E,\alpha}^N - \|u_2\|_{E,\alpha}^N) = N(c_m - c_\epsilon) \tag{51}$$

It follows from Eqs. (41) and (42) that there exists $\epsilon_0 \in (0, \epsilon_3)$ such that if $0 < \epsilon < \epsilon_0$, then

$$0 < c_m - c_\epsilon < \frac{1}{N}\left(\frac{N-\beta}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \tag{52}$$

For convenience, we denote

$$h_n = \frac{v_n}{\|v_n\|_{E,\alpha}},$$

$$h_0 = \frac{u_2}{(\|u_2\|_{E,\alpha}^N + N(c_m - c_\epsilon))^{1/N}}.$$

From Eq. (51) and $v_n \rightharpoonup u_2$ weakly in E, we have $h_n \rightharpoonup h_0$ weakly in E. Note that

$$\int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_0 |v_n|^{\frac{N}{N-1}})}{|x|^\beta}\mathrm{d}x = \int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_0 \|v_n\|_{E,\alpha}^{\frac{N}{N-1}} |h_n|^{\frac{N}{N-1}})}{|x|^\beta}\mathrm{d}x.$$

This together with Eqs. (51) and (52) yields

$$\lim_{n\to\infty}\|v_n\|_{E,\alpha}^{\frac{N}{N-1}}(1 - \|h_0\|_{E,\alpha}^N)^{\frac{1}{N-1}} < \left(1 - \frac{\beta}{N}\right)\alpha_N.$$

Then it follows from (Ref. [15], Lemma 2.1) and Lemma 1.1 that $\zeta(N, \alpha|u|^{N/(N-1)})/|x|^\beta$ is bounded in $L^q(\mathbb{R}^N)$ for some $q > 1$. By (H_3), we have

$$|f(x, v_n)| \leqslant a_1 |v_n|^{N-1} + a_2\zeta(N, \alpha_0 |v_n|^{\frac{N}{N-1}}).$$

Moreover by the continuous embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ for any $p \geqslant 1$, it follows that $f(x, v_n)/|x|^\beta$ is bounded in $L^{q_0}(\mathbb{R}^N)$ for some $q_0 > 1$. So by Lemma 1.2 and the Hölder inequality, we obtain

$$\left|\int_{\mathbb{R}^N} \frac{f(x, v_n)}{|x|^\beta}(v_n - u_2)\mathrm{d}x\right| \leqslant \left\|\frac{f(x, v_n)}{|x|^\beta}\right\|_{L^{q_0}(\mathbb{R}^N)} \|v_n - u_2\|_{L^{q_0'}(\mathbb{R}^N)},$$

where $1/q_0 + 1/q_0' = 1$. In view of Eq. (48) and the last inequality, by similar proofs of Eqs. (46) and (47), we get

$$\int_{\mathbb{R}^N}(|\nabla v_n|^{N-2}\nabla v_n \nabla(v_n - u_2) + V(x)|v_n|^{N-2}v_n(v_n - u_2))\mathrm{d}x - \alpha\|v_n\|_{L^p(\mathbb{R}^N)}^{N-p}\int_{\mathbb{R}^N}|v_n|^{p-2}v_n(v_n - u_2)\mathrm{d}x \to 0 \quad \text{as } n \to \infty \tag{53}$$

and

$$\int_{\mathbb{R}^N}(|\nabla u_2|^{N-2}\nabla u_2 \nabla(v_n - u_2) + V(x)|u_2|^{N-2}u_2(v_n - u_2))\mathrm{d}x - \alpha\|u_2\|_{L^p(\mathbb{R}^N)}^{N-p}\int_{\mathbb{R}^N}|u_2|^{p-2}u_2(v_n - u_2)\mathrm{d}x \to 0 \quad \text{as } n \to \infty \tag{54}$$

Subtracting Eq. (54) from Eq. (53) and applying (I) in Chapter 10 of Ref. [25], we can deduce that

$$\lim_{n\to\infty}\|v_n - u_2\|_{E,\alpha}^N = 0,$$

which together with Eq. (51) implies

$$c_m = c_\epsilon.$$

This yields a contradiction since $c_m > 0$ and $c_\epsilon < 0$.

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