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L^2-harmonic p-forms on submanifolds with finite total curvature

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Abstract: Let M be an n-dimensional complete submanifold with flat normal bundle in an $(n+l)$-dimensional sphere S^{n+l}. Let $H^p(L^2(M))$ be the space of all L^2-harmonic p-forms $(2\leqslant p\leqslant n-2)$ on M. Firstly, we show that $H^p(L^2(M))$ is trivial if the total curvature of M is less than a positive constant depending only on n. Secondly, we show that the dimension of $H^p(L^2(M))$ is finite provided the total curvature of M is finite.

Key words: Total curvature; L^2-harmonic p-form; Submanifold

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具有有限总曲率子流形的 L^2 调和 p 形式

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摘要:设 M 是 $n+l$ 维 S^{n+l} 球空间中具有法从平坦 n 维完备子流形,则 $H^p(L_2(M))$是 M 上 L^2 调和 $p(2\leqslant p\leqslant n-2)$ 形式空间.首先证明了如果 M 的总曲率小于一个正常数,则 $H^p(L^2(M))$是平凡的;其次证明了如果 M 的总曲率有限,则 $H^p(L^2(M))$是有限维的.

关键词:总曲率;L^2 调和 p 形式;子流形

0 Introduction

L^2-harmonic forms on submanifolds have been studied extensively in various ambient spaces during the last two decades. Many results demonstrated the fact that there is a close relation

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between the topology of the submanifold and the curvature according to the theory of L^2-harmonic forms. In Refs. [1-2], it was shown that a complete minimal hypersurface in Euclidean Space with the total scalar curvature condition has only one end. In 2008, Seo[3] improved the upper bound of the total scalar curvature which was given by Ni[2]. Later Seo[4] proved that if an n-dimensional complete minimal submanifold M in hyperbolic space has sufficiently small total scalar curvature, then M has only one end. In Ref. [5], Fu and Xu studied L^2-harmonic 1-forms on complete submanifolds in space forms and proved that a complete submanifold M^n ($n\geqslant 3$) with finite total curvature and some conditions on mean curvature must have finitely many ends. Furthermore, Cavalcante, Mirandola and Vitório[6] obtained that if a complete noncompact submanifold M^n ($n\geqslant 3$) in Cartan-Hadamard manifold has finite total curvature and the first eigenvalue with suitable lower bound, then the space of the L^2-harmonic 1-forms on M^n has finite dimension. Zhu and Fang[7] investigated complete noncompact submanifolds in a sphere and obtained a result which was an improvement of Fu and Xu's theorem on submanifolds in spheres. To be specific, they proved the following theorem.

Theorem A (Ref. [7]) Let M^n ($n\geqslant 3$) be an n-dimensional complete noncompact oriented manifold isometrically immersed in an ($n+l$)-dimensional sphere S^{n+l}. If the total curvature is finite, then the dimension of $H^1(L^2(M))$ is finite and there are finitely many non-parabolic ends on M. In 2015, Lin[8] studied L^2-harmonic p-forms on complete submanifolds M^n ($n\geqslant 3$) with flat normal bundles in Euclidean space and proved that if the total curvature of M^n is less than an explicit positive constant, then for any $2\leqslant p\leqslant n-2$, the space of the L^2-harmonic p-forms on M^n is trivial. Recently, Gan, Zhu and Fang[9] studied L^2-harmonic 2-forms on complete noncompact minimal hypersurface in spheres and proved the following result.

Theorem B (Ref. [9]) Let M^n ($n\geqslant 3$) be an n-dimensional complete noncompact minimal hypersurface isometrically immersed in an ($n+1$)-dimensional sphere S^{n+1}. There exists a positive constant $\delta(n)$ depending only on n such that if the total curvature is less than $\delta(n)$, then the second space of reduced L^2 cohomology of M is trivial.

Inspired by Li-Wang work[10] and the above results, in this paper, we study the space of L^2-harmonic p-forms on submanifold in spheres and prove the following vanishing and finiteness theorems.

Theorem 0.1 Let M be an n-dimensional ($n\geqslant 4$) complete noncompact submanifold with flat normal bundle in sphere S^{n+l}. There exists a positive constant $c(n)$ depending only on n such that if the total curvature is less than $c(n)$, then $H^p(L^2(M))=\{0\}$, $2\leqslant p\leqslant n-2$, where constant $c(n)$ is given by (8).

Theorem 0.2 Let M be an n-dimensional ($n\geqslant 4$) complete noncompact submanifold with flat normal bundle in sphere S^{n+l}. If the total curvature is finite and $2\leqslant p\leqslant n-2$, then the dimension of $H^p(L^2(M))$ is finite.

1 Preliminaries

Suppose M is an n-dimensional complete submanifold in an ($n+l$)-dimensional sphere S^{n+l}, A is the second fundamental form and H is the mean curvature vector of M. The traceless second fundamental form Φ is defined by

$$\Phi(X,Y)=A(X,Y)-\langle X,Y\rangle H,$$

for all vector fields X and Y, where $\langle\,,\rangle$ is the metric of M. Obviously

$$|\Phi|^2=|A|^2-n|H|^2.$$

We say M has finite total curvature if

$$\|\Phi\|_{L^n(M)}=\left(\int_M|\Phi|^n\right)^{\frac{1}{n}}<\infty.$$

$H^p(L^2(M))$ denotes the space of all L^2-harmonic p-forms on M. Choose local orthonormal frames $e_1,\cdots,e_{n+l}$ on S^{n+l} such that, restricted to M, $e_1,\cdots,e_n$ are tangent to M. Let $\omega_1,\cdots,\omega_{n+l}$ be the dual frames. We then have $\omega_\alpha=0$ for each α,

$n+1\leqslant\alpha\leqslant n+l$. From Cartan's Lemma, we have $\omega_{\alpha i}=h^{\alpha}_{ij}\omega_j$. The normal bundle of M is flat implies that there exists an orthonormal frame diagonalizing h^{α}_{ij} simultaneously.

Let us recall the following lemmas.

Lemma 1.1 Let M^n be an n-dimensional complete noncompact oriented submanifold in S^{n+l}, then

$$\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\leqslant\widetilde{C}\left[\int_M|\nabla f|^2+\int_M(|H|^2+1)f^2\right],$$

for each $f\in C_0^1(M)$, where $\widetilde{C}=n^2C_0$, C_0 depends only on n and H is the mean curvature vector of M in S^{n+l}.

Lemma 1.2 (Ref. [8, 11-12]) Let M^n be a complete submanifold with flat normal bundles in S^{n+l}, ω be a L^2-harmonic p-form ($2\leqslant p\leqslant n-2$) on M^n, then

$$|\omega|\Delta|\omega|\geqslant K_p|\nabla|\omega||^2+p(n-p)|\omega|^2+Q_p|\omega|^2,$$

where $Q_p=\inf_{i_1,\cdots,i_n}(h^{\alpha}_{i_1i_1}+\cdots+h^{\alpha}_{i_pi_p})(h^{\alpha}_{i_{p+1}i_{p+1}}+\cdots+h^{\alpha}_{i_ni_n})$, and $K_p=\frac{1}{n-p}$ if $2\leqslant p\leqslant\frac{n}{2}$, $K_p=\frac{1}{p}$ if $\frac{n}{2}\leqslant p\leqslant n-2$.

2 Proof of our main Theorems

Proof of Theorem 0.1 From the assumption, there exists an orthonormal frame diagonalizing h^{α}_{ij} simultaneously. Direct computation yields

$$\begin{aligned}
2\sum_{\alpha=n+1}^{n+l}(h^{\alpha}_{i_1i_1}+\cdots+h^{\alpha}_{i_pi_p})(h^{\alpha}_{i_{p+1}i_{p+1}}+\cdots+h^{\alpha}_{i_ni_n})=\\
\sum_{\alpha=n+1}^{n+l}(h^{\alpha}_{i_1i_1}+\cdots+h^{\alpha}_{i_ni_n})^2-\\
\sum_{\alpha=n+1}^{n+l}(h^{\alpha}_{i_1i_1}+\cdots+h^{\alpha}_{i_pi_p})^2-\\
\sum_{\alpha=n+1}^{n+l}(h^{\alpha}_{i_{p+1}i_{p+1}}+\cdots+h^{\alpha}_{i_ni_n})^2\geqslant\\
n^2|H|^2-\max\{p,n-p\}|A|^2=\\
\min\{p,n-p\}n|H|^2-\max\{p,n-p\}|\Phi|^2
\end{aligned}\tag{1}$$

Substituting Eq. (1) into Lemma 1.2, we have

$$\begin{aligned}
|\omega|\Delta|\omega|\geqslant K_p|\nabla|\omega||^2+\\
p(n-p)|\omega|^2+\\
\min\{p,n-p\}\frac{n}{2}|H|^2|\omega|^2-\\
\frac{1}{2}\max\{p,n-p\}|\Phi|^2|\omega|^2
\end{aligned}\tag{2}$$

This together with the condition $2\leqslant p\leqslant n-2$ yields

$$\begin{aligned}
|\omega|\Delta|\omega|\geqslant\frac{1}{n-2}|\nabla|\omega||^2+\\
2(n-2)|\omega|^2+n|H|^2|\omega|^2-\\
\frac{n-2}{2}|\Phi|^2|\omega|^2
\end{aligned}\tag{3}$$

Setting $\eta\in C_0^{\infty}(M)$, multiplying Eq. (3) by η^2 and integrating over M, we obtain

$$\begin{aligned}
\frac{n-2}{2}\int_M|\Phi|^2|\omega|^2\eta^2\geqslant\frac{n-1}{n-2}\int_M|\nabla|\omega||^2\eta^2+\\
2(n-2)\int_M|\omega|^2\eta^2+n\int_M|H|^2|\omega|^2\eta^2+\\
2\int_M\eta|\omega|\langle\nabla\eta,\nabla|\omega|\rangle
\end{aligned}\tag{4}$$

Combining the Hölder inequality with Lemma 1.1, we get

$$\begin{aligned}
\int_M|\Phi|^2|\omega|^2\eta^2\leqslant\\
\left(\int_M|\Phi|^n\right)^{\frac{2}{n}}\left(\int_M(|\omega|\eta)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\leqslant\\
\widetilde{C}\left(\int_M|\Phi|^n\right)^{\frac{2}{n}}\left[\int_M|\nabla(\eta|\omega|)|^2+\right.\\
\left.\int_M(|H|^2+1)|\omega|^2\eta^2\right]\leqslant\\
\widetilde{C}\left(\int_M|\Phi|^n\right)^{\frac{2}{n}}\left[\int_M(|\nabla|\omega||^2\eta^2+\right.\\
|\omega|^2|\nabla\eta|^2+2|\omega|\eta\langle\nabla\eta,\nabla|\omega|\rangle)+\\
\left.\int_M(|H|^2+1)|\omega|^2\eta^2\right]
\end{aligned}\tag{5}$$

Setting $E=\frac{n-2}{2}\widetilde{C}(\int_M|\Phi|^n)^{\frac{2}{n}}$ and using Eqs. (4) and (5) we have

$$\begin{aligned}
E\int_M|\omega|^2|\nabla\eta|^2+\\
2(E-1)\int_M|\omega|\eta\langle\nabla\eta,\nabla|\omega|\rangle\geqslant\\
\left(\frac{n-1}{n-2}-E\right)\int_M|\nabla|\omega||^2\eta^2+\\
[2(n-2)-E]\int_M|\omega|^2\eta^2+\\
(n-E)\int_M|H|^2|\omega|^2\eta^2
\end{aligned}\tag{6}$$

Using the Cauchy-Schwarz inequality in Eq. (6), we get

$$\begin{aligned}&(E+\frac{|E-1|}{\varepsilon})\int_M|\omega|^2|\nabla\eta|^2\geqslant\\&(\frac{n-1}{n-2}-E-|E-1|\varepsilon)\int_M|\nabla|\omega||^2\eta^2+\\&[2(n-2)-E]\int_M|\omega|^2\eta^2+\\&(n-E)\int_M|H|^2|\omega|^2\eta^2\end{aligned}\tag{7}$$

If

$$(\int_M|\Phi|^n)^{\frac{1}{n}}<\frac{2}{n-2}\sqrt{\frac{n-1}{2\widetilde{C}}}=c(n)\tag{8}$$

then

$$\frac{n-1}{n-2}-E>0.$$

Choosing sufficient small ε, we obtain

$$\frac{n-1}{n-2}-E-|E-1|\varepsilon>0,$$

$$n-E>0,\ 2(n-2)-E>0.$$

Let $\rho(x)$ be the geodesic distance on M from x_0 to x and $B_r(x_0)=\{x\in M:\rho(x)\leqslant r\}$ for some fixed point $x_0\in M$. Choose $\eta\in C_0^\infty(M)$ as

$$\eta=\begin{cases}1,\ \text{on } B_r(x_0),\\0,\ \text{on } M\backslash B_{2r}(x_0),\\|\nabla\eta|\leqslant\frac{2}{r},\ \text{on } B_{2r}(x_0)\backslash B_r(x_0),\end{cases}$$

and $0\leqslant\eta\leqslant1$. Substituting the above η into Eq. (7), we finally have

$$\begin{aligned}&\frac{4}{r^2}(E+\frac{|E-1|}{\varepsilon})\int_{B_{2r}(x_0)}|\omega|^2\geqslant\\&(\frac{n-1}{n-2}-E-|E-1|\varepsilon)\int_{B_r(x_0)}|\nabla|\omega||^2+\\&[2(n-2)-E]\int_{B_r(x_0)}|\omega|^2+\\&(n-E)\int_{B_r(x_0)}|H|^2|\omega|^2.\end{aligned}$$

Since $\int_M|\omega|^2<\infty$, by taking $r\to\infty$, we have $\nabla|\omega|=0$ and $\omega=0$. That is $H^p(L^2(M))=\{0\}$. This completes the proof of Theorem 0.1.

Proof of Theorem 0.2 Let $\omega\in H^p(L^2(M))$, $2\leqslant p\leqslant n-2$ and $\eta\in C_0^\infty(M\backslash B_r(x_0))$. Analogous to Eq. (7), using the proving method of Theorem 0.1 we deduce that

$$\begin{aligned}&(F+\frac{|F-1|}{\varepsilon})\int_{M\backslash B_r(x_0)}|\omega|^2|\nabla\eta|^2\geqslant\\&(\frac{n-1}{n-2}-F-|F-1|\varepsilon)\int_{M\backslash B_r(x_0)}|\nabla|\omega||^2\eta^2+\\&[2(n-2)-F]\int_{M\backslash B_r(x_0)}|\omega|^2\eta^2+\\&(n-F)\int_{M\backslash B_r(x_0)}|H|^2|\omega|^2\eta^2\end{aligned}\tag{9}$$

where $F=\frac{n-2}{2}\widetilde{C}(\int_{M\backslash B_r(x_0)}|\Phi|^n)^{\frac{2}{n}}$. The condition $(\int_M|\Phi|^n)^{\frac{1}{n}}<\infty$ implies that there is a decreasing positive function $\varepsilon(r)$ satisfying

$$\lim_{r\to\infty}\varepsilon(r)=0,\ (\int_{M\backslash B_r(x_0)}|\Phi|^n)^{\frac{2}{n}}<\varepsilon(r).$$

Thus we can choose $r=r_0>0$ such that

$$\frac{n-1}{n-2}-F=\frac{n-1}{n-2}-\frac{n-2}{2}\widetilde{C}(\int_{M\backslash B_{r_0}(x_0)}|\Phi|^n)^{\frac{2}{n}}>0.$$

Choosing sufficient small ε, we get

$$\frac{n-1}{n-2}-F-|F-1|\varepsilon>0.$$

This together with Eq. (9) yields that

$$\begin{aligned}&\int_{M\backslash B_{r_0}(x_0)}|\nabla|\omega||^2\eta^2\leqslant\\&\frac{F+\frac{|F-1|}{\varepsilon}}{\frac{n-1}{n-2}-F-|F-1|\varepsilon}\int_{M\backslash B_{r_0}(x_0)}|\omega|^2|\nabla\eta|^2=\\&C_1\int_{M\backslash B_{r_0}(x_0)}|\omega|^2|\nabla\eta|^2\end{aligned}\tag{10}$$

$$\begin{aligned}&\int_{M\backslash B_{r_0}(x_0)}|\omega|^2\eta^2\leqslant\\&\frac{F+\frac{|F-1|}{\varepsilon}}{2(n-2)-F}\int_{M\backslash B_{r_0}(x_0)}|\omega|^2|\nabla\eta|^2\leqslant\\&C_1\int_{M\backslash B_{r_0}(x_0)}|\omega|^2|\nabla\eta|^2\end{aligned}\tag{11}$$

$$\begin{aligned}&\int_{M\backslash B_{r_0}(x_0)}|H|^2|\omega|^2\eta^2\leqslant\\&\frac{F+\frac{|F-1|}{\varepsilon}}{n-F}\int_{M\backslash B_{r_0}(x_0)}|\omega|^2|\nabla\eta|^2\leqslant\\&C_1\int_{M\backslash B_{r_0}(x_0)}|\omega|^2|\nabla\eta|^2\end{aligned}\tag{12}$$

where the positive constant C_1 depends only on n.

Applying Lemma 1.1 to $\eta|\omega|$ and combining Eqs. (10), (11) and (12), we obtain

$$\int_{M\backslash B_{r_0}(x_0)}(\eta|\omega|)^{\frac{2n}{n-2}})^{\frac{n-2}{n}}\leqslant$$
$$\widetilde{C}\int_{M\backslash B_{r_0}(x_0)}[|\nabla|\omega||^2\eta^2+|\omega|^2|\nabla\eta|^2+$$
$$2|\omega|\eta\langle\nabla\eta,\nabla|\omega|\rangle+(|H|^2+1)|\omega|^2\eta^2]\leqslant$$
$$\widetilde{C}\int_{M\backslash B_{r_0}(x_0)}[2|\nabla|\omega||^2\eta^2+2|\omega|^2|\nabla\eta|^2+$$
$$(|H|^2+1)|\omega|^2\eta^2]\leqslant$$
$$C_2\int_{M\backslash B_{r_0}(x_0)}|\omega|^2|\nabla\eta|^2 \tag{13}$$

where positive constant C_2 depends only on n.

Choose $\eta\in C_0^\infty(M\backslash B_{r_0}(x_0))$ as

$$\eta=\begin{cases}0, \text{ on } B_{r_0}(x_0),\\ \rho(x)-r_0, \text{ on } B_{r_0+1}(x_0)\backslash B_{r_0}(x_0),\\ 1, \text{ on } B_r(x_0)\backslash B_{r_0+1}(x_0),\\ \dfrac{2r-\rho(x)}{r}, \text{ on } B_{2r}(x_0)\backslash B_r(x_0),\\ 0, \text{ on } M\backslash B_{2r}(x_0),\end{cases}$$

where $\rho(x)$ is the geodesic distance on M from x_0 to x and $r>r_0+1$. Substituting η into Eq. (13) it yields that

$$\int_{B_r(x_0)\backslash B_{r_0+1}(x_0)}(|\omega|^{\frac{2n}{n-2}})^{\frac{n-2}{n}}\leqslant$$
$$C_2\int_{B_{r_0+1}(x_0)\backslash B_{r_0}(x_0)}|\omega|^2+\frac{C_2}{r^2}\int_{B_{2r}(x_0)\backslash B_r(x_0)}|\omega|^2 \tag{14}$$

Since $|\omega|\in L^2(M)$, letting $r\to\infty$, we conclude that

$$\int_{B_r(x_0)\backslash B_{r_0+1}(x_0)}(|\omega|^{\frac{2n}{n-2}})^{\frac{n-2}{n}}\leqslant$$
$$C_2\int_{\{}B_{r_0+1}(x_0)\backslash B_{r_0}(x_0)\}|\omega|^2 \tag{15}$$

On the other hand, the Hölder inequality asserts that

$$\int_{B_{r_0+2}(x_0)\backslash B_{r_0+1}(x_0)}|\omega|^2\leqslant$$
$$\mathrm{vol}(B_{r_0+2}(x_0)))^{\frac{2}{n}}\int_{B_{r_0+2}(x_0)\backslash B_{r_0+1}(x_0)}(|\omega|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \tag{16}$$

From Eqs. (15) and (16), we conclude that there exists a constant $C_3>0$ depending on $\mathrm{vol}(B_{r_0+2}(x_0))$ and n such that

$$\int_{B_{r_0+2}(x_0)}|\omega|^2\leqslant C_3\int_{B_{r_0+1}(x_0)}|\omega|^2 \tag{17}$$

Fix a point $x\in M$ and take $\tau\in C_0^1(B_1(x))$. Multiplying Eq. (3) by $|\omega|^{q-2}\tau^2$ with $q>2$ and integrating by parts on $B_1(x)$, we obtain

$$-2\int_{B_1(x)}\tau|\omega|^{q-1}\langle\nabla\tau,\nabla|\omega|\rangle+$$
$$\frac{n-2}{2}\int_{B_1(x)}|\Phi|^2|\omega|^q\tau^2\geqslant$$
$$(\frac{1}{n-2}+q-1)\int_{B_1(x)}|\omega|^{q-2}|\nabla|\omega||^2\tau^2+$$
$$2(n-2)\int_{B_1(x)}|\omega|^q\tau^2+n\int_{B_1(x)}|H|^2|\omega|^q\tau^2 \tag{18}$$

By using the Cauchy-Schwarz inequality, we have

$$-2\int_{B_1(x)}\tau|\omega|^{q-1}\langle\nabla\tau,\nabla|\omega|\rangle\leqslant$$
$$\frac{1}{n-2}\int_{B_1(x)}|\omega|^{q-2}|\nabla|\omega||^2\tau^2+$$
$$(n-2)\int_{B_1(x)}|\omega|^q|\nabla\tau|^2 \tag{19}$$

It follows from Eqs. (18) and (19) that

$$(n-2)\int_{B_1(x)}|\omega|^q|\nabla\tau|^2+$$
$$\frac{n-2}{2}\int_{B_1(x)}|\Phi|^2|\omega|^q\tau^2\geqslant$$
$$(q-1)\int_{B_1(x)}|\omega|^{q-2}|\nabla|\omega||^2\tau^2+$$
$$2(n-2)\int_{B_1(x)}|\omega|^q\tau^2+n\int_{B_1(x)}|H|^2|\omega|^q\tau^2 \tag{20}$$

On the other hand, setting $f\in C_0^1(B_1(x))$, similar to Lemma 1.1, we have

$$(\int_{B_1(x)}|f|^{\frac{2n}{n-2}})^{\frac{n-2}{n}}\leqslant$$
$$\widetilde{C}[\int_{B_1(x)}|\nabla f|^2+\int_{B_1(x)}(|H|^2+1)f^2] \tag{21}$$

Applying Eq. (21) to $\tau|\omega|^{\frac{q}{2}}$, we obtain

$$(\int_{B_1(x)}(\tau^2|\omega|^q)^{\frac{n}{n-2}})^{\frac{n-2}{n}}\leqslant$$
$$\widetilde{C}\int_{B_1(x)}|\nabla(\tau|\omega|^{\frac{q}{2}})|^2+$$
$$\widetilde{C}\int_{B_1(x)}(|H|^2+1)\tau^2|\omega|^q\leqslant$$

$$2\widetilde{C}\int_{B_1(x)}|\nabla\tau|^2|\omega|^q +$$

$$\frac{q^2}{2}\widetilde{C}\int_{B_1(x)}\tau^2|\omega|^{q-2}|\nabla|\omega||^2 +$$

$$\widetilde{C}\int_{B_1(x)}(|H|^2+1)\tau^2|\omega|^q \qquad (22)$$

Inequalities (22) and (20) imply that

$$\left(\int_{B_1(x)}(\tau^2|\omega|^q)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \leqslant$$

$$2\widetilde{C}\int_{B_1(x)}|\nabla\tau|^2|\omega|^q +$$

$$\frac{q^2}{2(q-1)}\widetilde{C}\int_{B_1(x)}[(n-2)|\nabla\tau|^2 +$$

$$\frac{n-2}{2}|\Phi|^2\tau^2]|\omega|^q -$$

$$\frac{q^2}{2(q-1)}\widetilde{C}\int_{B_1(x)}[2(n-2) +$$

$$n|H|^2]|\omega|^q\tau^2 +$$

$$\widetilde{C}\int_{B_1(x)}(|H|^2+1)\tau^2|\omega|^q \leqslant$$

$$qC_4\int_{B_1(x)}(|\nabla\tau|^2+|\Phi|^2\tau^2)|\omega|^q \qquad (23)$$

where C_4 is a positive constant depending only on n. Let $q_k=\frac{2n^k}{(n-2)^k}$ and $r_k=\frac{1}{2}+\frac{1}{2^{k+1}}$ for an integer $k\geqslant 0$. Choose $\tau_k\in C_0^\infty(B_{r_k}(x))$ such that $\tau_k=1$ on $B_{r_{k+1}}(x)$ and $|\nabla\tau_k|\leqslant 2^{k+3}$. Replacing q and τ in Eq. (23) by q_k and τ_k respectively, we obtain

$$\left(\int_{\{}B_{r_{k+1}}(x)\}|\omega|^{q_{k+1}}\right)^{\frac{1}{q_{k+1}}} \leqslant$$

$$[q_kC_4(4^{k+3}+\sup_{B_1(x)}|\Phi|^2)]^{\frac{1}{q_k}}\left(\int_{B_{r_k}(x)}|\omega|^{q_k}\right)^{\frac{1}{q_k}} \qquad (24)$$

Apply the Morse interation to $|\omega|$ via (24), we conclude that

$$\|\omega\|^2_{L^\infty(B_{\frac{1}{2}}(x))} \leqslant C_5\int_{B_1(x)}|\omega|^2,$$

where C_5 is a positive constant depending only on n. Obviously

$$|\omega(x)|^2 \leqslant C_5\int_{B_1(x)}|\omega|^2 \qquad (25)$$

Choose $x\in\overline{B_{r_0+1}(x_0)}$ such that

$$|\omega(x)|^2=\|\omega\|^2_{L^\infty(B_{r_0+1}(x_0))}.$$

This together with Eq. (25) yields that

$$\|\omega\|^2_{L^\infty(B_{r_0+1}(x_0))}=|\omega(x)|^2 \leqslant$$

$$C_5\int_{B_1(x)}|\omega|^2 \leqslant C_5\int_{B_{r_0+2}(x_0)}|\omega|^2 \qquad (26)$$

This together with Eq. (17) implies that there exists a positive constant C_6 depending on n and $\mathrm{vol}(B_{r_0+2}(x_0))$, such that

$$\sup_{B_{r_0+1}(x_0)}|\omega|^2 \leqslant C_6\int_{B_{r_0+1}(x_0)}|\omega|^2 \qquad (27)$$

Let φ be a finite dimensional subspace of $H^p(L^2(M))$. Lemma 11 in Ref. [13] implies that there exits $\omega\in\varphi$ such that

$$\frac{\dim\varphi}{\mathrm{vol}(B_{r_0+1}(x_0))}\int_{B_{r_0+1}(x_0)}|\omega|^2 \leqslant$$

$$|\{\tbinom{n}{p},\dim\varphi\sup_{B_{r_0+1}(x_0)}|\omega|^2.$$

This together with (27) yields $\mathrm{dim}\,\varphi\leqslant C_7$, where C_7 depends on n and $\mathrm{vol}(B_{r_0+1}(x_0))$. Hence $\dim H^p(L^2(M))<\infty$, which completes the proof of Theorem 0.2.

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References

[1] YUN G. Total scalar curvature andL^2-harmonic 1-forms on a minimal hypersurface in Euclidean space [J]. Geom. Dedicata, 2002, 89: 135-141.

[2] NI L. Gap theorems for minimal submanifolds in R^{n+1}[J]. Commun. Anal. Geom., 2001,9: 641-656.

[3] SEO K. Minimal submanifolds with small total scalar curvature in Euclidean space [J]. Kodai Math. J., 2008, 31: 113-119.

[4] SEO K. Rigidity of minimal submanifolds in hyperbolic space[J]. Arch. Math. 2010, 94: 173-181.

[5] FU H P, XU H W. Total curvature and L^2-harmonic 1-forms on complete submanifolds in space forms[J]. Geom. Dedicata, 2010, 144: 129-140.

[6] CAVALCANTE M P, MIRANDOLA H, VITÓRIO F. L^2-harmonic 1-forms on submanifolds with nite total curvature [J]. J. Geom. Anal., 2014, 24: 205-222.

[7] ZHU P, FANG SW. Finiteness of non-parabolic ends on submanifolds in spheres [J]. Ann. Global Anal. Geom., 2014, 46: 187-196.

[8] LIN H Z. Vanishing theorems for harmonic forms on complete submanifolds in Euclidean space [J]. J. Math. Anal. Appl., 2015, 425: 774-787.

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