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The Erdős-Sós conjecture for 2-center spiders

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Abstract: The Erdős-Sós Conjecture states that if G is a graph with average degree more than $k-2$, then G contains every tree on k vertices. A spider can be seen as a tree with at most one vertex of degree more than two. Fan, Hong, and Liu proved that the conjecture holds for spiders. In this note, we define a 2-center spider as a tree with at most two adjacent vertices of degree more than two and show that the Erdős-Sós Conjecture holds for 2-center spiders with legs of lengths at most two adjacent vertices of degree more than 2 as 2-center spider. We prove that if G is a graph on n vertices with average degree more than $k-2$, then G contains every 2-center spider with k vertices, where length of 2-center spider′ legs is no more than 2.

Key words: Erdős-Sós conjecture, tree, spider, 2-center spider

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关于2-中心蜘蛛树的 Erdős-Sós 猜想

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摘要:Erdős-Sós 猜想:如果图 G 平均度大于 k—2,则 G 包含任一 k 个顶点的数. 蜘蛛树是指最多只有一个点度超过 2 的树. 范更华、洪艳梅和刘清海证明了该猜想对所有蜘蛛树成立. 本文我们定义 2 中心蜘蛛树为至多两个相邻点度超过 2 的树并且证明了 Erdős-Sós 猜想对腿长至多为 2 的 2 中心蜘蛛树都成立.

关键词:Erdős-Sós 猜想; 树;蜘蛛树;2-中心蜘蛛树

0 Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively, and $e(G)=|E(G)|$. The following conjecture is well known as the Erdős-Sós conjecture.

Conjecture 0.1 (Erdős-Sós conjecture[1]). Every graph G with $|E(G)|>(k-2)|V|/2$

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contains every tree of order k as a subgraph.

There are many partial results in the study of the conjecture, especially for special family of trees on k vertices, such as:

(a) path (Erdős and Gallai[2]);

(b) spiders (A spider is a tree with at most one vertex of degree more than two[3-5];

(c) caterpillars (A caterpillar is a tree in which the vertices of degree more than one induce a path. Perle[6-7]);

(d) trees of diameter at most four (McLennan[8]), and (e) trees with a vertex joined to at least $\lfloor \frac{k}{2} \rfloor - 1$ vertices of degree one[9].

In this note, we consider a special family of spider-like trees: 2-center spider. A leaf of a graph is a vertex of degree one, and the neighbor of a leaf is called the support vertex of it. Let G be a graph. We use $L(G)$ and $S(G)$ denote the set of leaves of G and the set of support vertices of and $S(G)$ to denote the set of leaves of G and the set of support vertices of G, respectively. The diameter of G is the length of a longest path in G. As we know, a spider is a tree with at most one vertex of degree at great than two. A 2-center spider is a tree with at most two adjacent vertices of degrees greater than two. Let T be a 2-center spider. If $\Delta(T) \leqslant 2$ then any vertex of T can be seen as the center; if $\Delta(T) \geqslant 3$, we call the vertices of degrees more than two the centers. The shortest path joining a leaf to the centers is called a leg of T.

Theorem 0.1 Every graph G with $|E(G)| > (k-2)|V|/2$ contains every 2-center spider of order k and with legs of lengths no more than 2.

Here is some notation which will be used in the proof. Let G be a graph. For $v \in V(G)$, we write $N_G(v)$ for the set of neighbors of v in G, and $N_G^i(v)$ for the set of vertices with distance i from v in G. For $S, T \subseteq V(G)$, write $E_G(S, T)$ for the set of edges with one end in S and the other in T, write $G[S]$ be the subgraph induced by S. We give the proof of Theorem 0.1 in the next section. Section 2 will give some remarks and discussions.

1 Proof of Theorem 0.1

We first give two useful observations.

Lemma 1.1 Let T be a tree and G be a graph. Suppose $v_1, v_2, w_1 \in L(T)$ and $v, w \in S(T)$ such that $v_1v, v_2v, w_1w \in E(T)$. Let $T' = T - vv_1 + v_1v_2$. If T can be embedded in G and there is a perfect matching between $\{v_1, v_2\}$ and $\{w, w_1\}$ in G, then G contains a copy of T' as a subgraph.

Proof Let M be a matching between $\{v_1, v_2\}$ and $\{w, w_1\}$. If $M = \{v_1w, v_2w_1\}$, then $T - \{vv_1, ww_1\} + \{wv_1, v_2w_1\}$ is a subgraph of G isomorphic to T'. If $M = \{v_1w_1, v_2w\}$ then $T - \{vv_2, ww_1\} + \{wv_2, v_1w_1\}$ is a subgraph of G isomorphic to T'.

Lemma 1.2 Suppose G is a graph. Let $X = \{x_1, x_2\} \subseteq V(G)$ and $Y = \bigcup_{i=1}^{m} \{y_i, z_i\} \subseteq V(G)$. If $|E_G(X, Y)| > 2m$, then there exists a pair $\{y_i, z_i\}$ such that there is a perfect matching between X and $\{y_i, z_i\}$ in G.

Proof Since $|E_G(X, Y)| > 2m$, there must be a pair $Y_i = \{y_i, z_i\}$ so that $|E_G(X, Y_i)| \geqslant 3$. Thus the bipartite subgraph with partitions X, Y_i contains a perfect matching.

Lemma 1.3[10] If G is a graph on n vertices with $e(G) > \frac{n(k-2)}{2}$, then there is a subgraph $H \subseteq G$ with $e(H) > \frac{|V(H)|(k-2)}{2}$ and $\delta(H) \geqslant \lfloor \frac{k}{2} \rfloor$.

Now we are ready to give the proof of Theorem 0.1.

Proof of Theorem 0.1 By Lemma 1.3, we may assume $\delta(G) \geqslant \frac{k}{2}$. By the result of Erdős and Gallai (see (a)) and the result of Fan, Hong and Liu (see (b)), it is sufficient to show the theorem for T with two adjacent vertices of degrees more than two. Denote by a and b the two centers of T, respectively. Let $L_T(a) = N_T(a) \cap L(T)$ and

$L_T(b)=N_T(b)\cap L(T)$. Without loss of generality, assume $|L_T(a)|\geqslant|L_T(b)|$. Let $L_T^2(a)=N_T^2(a)\setminus N_T(b)$ and $L_T^2(b)=N_T^2(b)\setminus N_T(a)$. Let $S(L_T^2(a))=N_T(a)\setminus(L_T(a)\cup\{b\})$ and $S(L_T^2(b))=N_T(b)\setminus(L_T(b)\cup\{a\})$. Since the length of each leg of T is at most 2, we have

$$V(T)=L_T^2(a)\cup L_T^2(b)\cup S(L_T^2(a))\cup S(L_T^2(b))\cup L_T(a)\cup L_T(b)\cup\{a,b\}.$$

We use induction on the number of $|L_T^2(a)|$. If $|L_T^2(a)|=0$, then T has diameter at most 4. So the base case follows from the result of McLennan (see (d)). Now suppose $|L_T^2(a)|=\ell\geqslant 1$ and the result holds for all 2-center spider trees T' with a center c having $|L_{T'}^2(c)|<\ell$. Suppose to the contrary that G does not contain T as a subgraph. Choose $z\in L_T^2(a)$. Let $T'=T-z+za$. Then $L_{T'}^2(a)=L_T^2(a)\setminus\{z\}$ and $L_{T'}(a)=L_T(a)\cup\{z,s(z)\}$, where $s(z)$ is the support vertex of z in T. Then $|L_{T'}^2(a)|=\ell-1$. By induction hypothesis, T' can be embedded in G.

Claim 1.1 (i) $L_{T'}(a)$ is an independent set in G;

(ii) $N_G(L_{T'}(a))\subseteq V(T)$.

Proof of the claim 1.1 (i) Suppose to the contrary that there are x, $y\in L_{T'}(a)$ such that $xy\in E(G)$. Without loss of generality, assume $y\neq z$. Then we have $T\cong T'-\{ay, az\}+xy\subseteq G$, a contradiction.

(ii) Suppose to the contrary that there is a leaf $x\in L_{T'}(a)$ such that x has a neighbor $h\notin V(T')$. If $x=z$ then $T\cong T'-s(z)a+zh$ contained in G, otherwise $T\cong T'-za+xh$ is a subgraph of G, a contradiction.

Let $p=|L_T(a)|$ and $q=|L_T(b)|$. Note that $p\geqslant q$ and $|L_{T'}(a)|=p+2$ and $L_{T'}(b)=L_T(b)$.

Case 1.1 $p\geqslant 2$.

Let x,y be two leaves in $L_{T'}(a)$. By Claim 1.1, $L_{T'}(a)$ is an independent set. Let $U=L_{T'}^2(a)\cup L_{T'}^2(b)$ and W be the set of support vertices of U in T'. Then $T'[U\cup W]$ is a matching in T'. Denote $m=|U|$. We have

$$m=|U|=|W|=\frac{|V(T)|-|L_{T'}(a)|-|L_{T'}(b)|-|\{a,b\}|}{2}=\frac{k-p-q-4}{2}.$$

By Claim 1.1(ii), $N_G(x)$, $N_G(y)\subseteq V(T')$.

Case 1.1.1 $x\in N_G(b)\cap N_G(L_T(b))$.

We claim that $y\notin N_G(b)\cup N_G(L_T(b))$. If not, then there must be a leaf $b'\in L_T(b)$ such that there is a matching M between $\{x, y\}$ and $\{b, b'\}$. By Lemma 1.1, there is a copy of $T'-ax+xy$ which can be embedded in G. This is impossible since $T\cong T'-ax+xy$. Since $\delta(G)\geqslant\frac{k}{2}$, $|E_G(x, U\cup W)|\geqslant\frac{k}{2}-2-q$ and $|E_G(y, U\cup W)|\geqslant\frac{k}{2}-1$. So, $|E_G(\{x,y\}, U\cup W)|\geqslant k-q-3>2m$. By Lemma 1.2, there is a matching between $\{x,y\}$ and a pair $\{u,w\}$ with $uw\in E(T'[U\cup W])$. By Lemma 1.1, G contains a copy of T, a contradiction.

Case 1.1.2 $x\in N_G(b)$ but $x\notin N_G(L_T(b))$.

With a similar reason with Case 1.1.1, we have $y\notin N_G(L_T(b))$. Since $\delta(G)\geqslant\frac{k}{2}$, we have $|E_G(x, U\cup W)|\geqslant\frac{k}{2}-2$ and $|E_G(y, U\cup W)|\geqslant\frac{k}{2}-2$. Therefore, $|E_G(\{x,y\}, U\cup W)|\geqslant k-4>2m$. By Lemma 1.2, there is a matching between $\{x,y\}$ and a pair $\{u,w\}$ with $uw\in E(T'[U\cup W])$. By Lemma 1.1, G contains a copy of T, a contradiction.

Case 1.1.3 $x\notin N_G(b)\cup N_G(L_{T'}(b))$.

Since $\delta(G)\geqslant\frac{k}{2}$, we have $|E_G(x, U\cup W)|\geqslant\frac{k}{2}-1$ and $|E_G(y, U\cup W)|\geqslant\frac{k}{2}-2-q$. Therefore, $|E_G(\{x,y\}, U\cup W)|\geqslant k-3-q>2m$. By Lemma 1.2, there is a matching between $\{x,y\}$ and a pair $\{u,w\}$ with $uw\in E(T'[U\cup W])$. By Lemma 1.1, G contains a copy of T, a contradiction.

Case 1.1.4 $x\notin N_G(b)$ but $x\in N_G(L_T(b))$.

Let $x\in N_G(b')$, where $b'\in L_T(b)$. We claim that $y \notin N_G(b)$. If not, there is a matching M between $\{x, y\}$ and $\{b, b'\}$. By Lemma 1.1, there is a copy of $T \cong T' - ay + by + xb'$ which can be embedded in G. Since $\delta(G) \geqslant \frac{k}{2}$, we have $|E_G(x, U \cup W)| \geqslant \frac{k}{2} - 1 - q$ and $|E_G(y, U \cup W)| \geqslant \frac{k}{2} - 1 - q$. Therefore, $|E_G(\{x, y\}, U \cup W)| \geqslant k - 2 - 2q > 2m$. By Lemma 1.2, there is a matching between $\{x, y\}$ and a pair $\{u, w\}$ with $uw \in E(T'[U \cup W])$. By Lemma 1.1, G contains a copy of T, a contradiction.

Case 1.2 $p = 1$.

Case 1.2.1 $q = 0$

Then $|E_G(\{x, y\}, U \cup W)| \geqslant 2 \cdot (\frac{k}{2} - 2) = k - 4 > k - 5 = 2m$. By Lemma 1.2, there is a matching between $\{x, y\}$ and a pair $\{u, w\}$ with $uw \in E(T'[U \cup W])$. We again get a contradiction by Lemma 1.1 with the same reason as Case 1.1.1.

Case 1.2.2 $q = 1$

Let $L_T(b) = \{b'\}$. Set $U' = U \cup \{b'\}$ and $W' = W \cup \{b\}$. Then $|U'| = |W'| = m + 1 = \frac{k-6}{2} + 1 = \frac{k-4}{2}$. Since $\delta(G) \geqslant \frac{k}{2}$, $|E_G(\{x, y\}, U' \cup W')| \geqslant 2 \cdot (\frac{k}{2} - 1) = k - 2 > k - 4 = 2(m + 1)$. By Lemma 1.2, there is a matching between $\{x, y\}$ and a pair $\{u, w\}$ with $u \in U'$ and $w \in W'$. Again we get a contradiction by Lemma 1.1 with the same reason as Case 1.1.1.

Case 1.3 $p = q = 0$.

In this case, we may assume $|L_T^2(a)| \leqslant |L_T^2(b)|$ by the symmetry of a and b (otherwise, we may choose $z \in L_T^2(b)$ instead of $z \in L_T^2(a)$). Then $|L_T^2(a)| \leqslant \lfloor \frac{k-2}{4} \rfloor$. Without loss of generality, assume $x = z$. Then y is the support vertex of x in T. Since $\delta(G) \geqslant \frac{k}{2}$, we have $|E_G(x, U \cup W)| \geqslant \frac{k}{2} - 2$ and $|E_G(y, U \cup W)| \geqslant \frac{k}{2} - 2$. We claim that $|E_G(x, U \cup W)| = |E_G(y, U \cup W)| = \frac{k}{2} - 2$. Otherwise, at least one of x, y has neighbors more than $\frac{k}{2} - 2$ in $U \cup W$, then $|E_G(\{x, y\}, U \cup W)| > 2 \cdot (\frac{k}{2} - 2) = k - 4 = 2m$. Similar as Case 1.1.1, by Lemmas 1.1 and 1.1, we get a contradiction. The claim also implies that $x, y \in N_G(b)$.

Claim 1.2 For every edge $e = uw \in E(T'[U \cup W])$ with $u \in L_T^2(a)$, we have either $V(e) \cap N_G(x) \neq \emptyset$ or $V(e) \cap N_G(b) \neq \emptyset$.

Proof of the claim 1.2 If not, we claim that $uy, wy \in E(G)$. Otherwise, set $U' = U \setminus \{u\}$ and $W' = W \setminus \{w\}$, then $|U'| = m - 1 = \frac{k-4}{2} - 1 = \frac{k-6}{2}$. But $|E_G(x, U' \cup W')| = \frac{k}{2} - 2$ and $|E_G(y, U' \cup W')| = \frac{k}{2} - 2 - 1 = \frac{k}{2} - 3$. So, $|E_G(\{x, y\}, U' \cup W')| = k - 5 > 2|U'|$. Therefore, by Lemmas 1.2 and 1.1, we can get a contradiction similar as Case 1.1.1. Now set $T'' = T' - uw + yu$. Then $T'' \cong T'$ and $L_{T''}(a) = \{x, w\}$ but $wb \notin E(G)$. We reset $U' = U$ and $W' = (W \setminus \{w\}) \cup \{y\}$. Then $|U'| = |W'| = m = \frac{k-4}{2}$. But $|E_G(x, U' \cup W')| = \frac{k}{2} - 2$ and $|E_G(w, U' \cup W')| \geqslant \frac{k}{2} - 1$. So, $|E_G(\{x, w\}, U' \cup W')| = k - 3 > 2m$. Again by Lemmas 1.2 and 1.1, we can get a contradiction similar as Case 1.1.1.

Now we claim that we can find a copy of T in G with centers x and b. Since $|E_G(x, U \cup W)| \geqslant \frac{k-4}{2}$, there are at least $\lceil \frac{k-4}{4} \rceil$ edges e of $E(T'[U \cup W])$ such that $V(e) \cap N_G(x) \neq \emptyset$. Since $|L_T^2(a)| \leqslant \lfloor \frac{k-2}{4} \rfloor \backslash \lceil \frac{k-4}{4} \rceil$, we can choose $|L_T^2(a)|$ edges in $E(T'[U \cup W])$ connected to x (we prefer to choose the edges with one end in $L_{T'}^2(a)$). Note that $|L_{T'}^2(a)| < |L_T^2(a)|$, for the rest edges with one end in $L^2_{T'}(a)$, by Claim 1.2 and our choice of edges connected to x, this edge must have one end adjacent to b, and so it can be seen as an edge of a leg of length 2 connected to b. For the other edges of $E(T'[U \cup W])$, each edge has one end in $L^2_T(b)$, it is still an edge of a leg connected to b. Moreover, the path yab is also a leg of length 2 connected to b. Clearly, the resulting tree is isomorphic to T, as desired.

This completes the proof of the theorem.

2 Remarks and Discussions

In the study of the Erdős-Sós Conjecture, the spiders have been well studied and verified completely by Ref. [4] recently, but the first step to attack the special family is to show the Erdős-Sós Conjecture holds for spiders with legs of length at most two (Woźniak[11]). In this note, we initially study the 2-center spiders, a generalization of spider, and prove that the Erdős-Sós Conjecture holds for 2-center spiders with legs of length at most two. We hope that one can show the Erdős-Sós Conjecture holds for 2-spiders with no restriction on the length of legs in the near future.

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