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The stable subgroups of S_n acting on $\mathscr{M}_{0,n}$

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Abstract: Considering the action of the symmetric group S_n on $\mathscr{M}_{0,n}$, all the possible stable subgroups were obtained.

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1 Introduction

The moduli space of the Riemann sphere with n-marked points is

$$\mathscr{M}_{0,n} = \{(x_1,\cdots,x_n) \in \Pi\mathbb{CP}^1 \mid x_i \neq x_j, \forall 1 \leqslant i \neq j \leqslant n\}/\mathrm{PGL}_2(\mathbb{C}).$$

The symmetric group S_n naturally acts on $\mathscr{M}_{0,n}$. For any $\sigma \in S_n$ and any $[x_1,\cdots,x_n] \in \mathscr{M}_{0,n}$, we have

$$\sigma \cdot [x_1,\cdots,x_n] = [x_{\sigma(1)},\cdots,x_{\sigma(n)}].$$

Ref. [1] investigated the locus with nontrivial stable subgroups, and proved that the stable subgroups must be cyclic. However, we find examples with the stable subgroups being cyclic groups, dihedral groups, A_4, S_4, or A_5. In this paper, we study stable subgroups on $\mathscr{M}_{0,n}$, $n \geqslant 4$, and obtain all types of stable subgroups. For the convenience of description, we introduce the following notations. We write $\zeta_d, d \geqslant 1$, to denote the primitive d-th root of unity in $\mathbb{C}$, and write $\overline{A}$ to denote the image of $A \in \mathrm{GL}_2(\mathbb{C})$ in the projective general linear group $\mathrm{PGL}_2(\mathbb{C})$. Let $x \in \mathscr{M}_{0,n}$, then the stable subgroup of x is

$$\mathrm{Stab}(x) = \{\sigma \in S_n \mid \sigma \cdot x = x\}.$$

We first prove that the stable subgroup is isomorphic to a finite subgroup G of $\mathrm{PGL}_2(\mathbb{C})$, and then obtain the types of p-subgroups by using the properties of elements in G, and then obtain the types of finite subgroups in $\mathrm{PGL}_2(\mathbb{C})$. Then we obtain the classification of stable subgroups (see Theorem 2.1). Finally, we further discuss stable subgroups for a more accurate description (see Proposition 3.1), and show the existence of these possible types by some examples (see Example 3.1).

2 Finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$

Theorem 2.1 Let $x \in \mathscr{M}_{0,n}$, then $\mathrm{Stab}(x)$ is isomorphic to a cyclic group, a dihedral group, A_4, S_4 or A_5.

Let $x = [a_1,\cdots,a_n] \in \mathscr{M}_{0,n}$ and $(a_1,a_2,a_3) = (0,\infty,1)$. Then for every $\sigma \in \mathrm{Stab}(x)$, there exists a unique $\overline{A} \in \mathrm{PGL}_2(\mathbb{C})$ such that $\sigma \cdot (a_1,\cdots,a_n) = \overline{A} \cdot (a_1,\cdots,a_n)$. So we can define a map $\Phi: \mathrm{Stab}(x) \to \mathrm{PGL}_2(\mathbb{C})$. Obviously, Φ is a group homomorphism and it is injective. Therefore, we obtain the following conclusion:

Proposition 2.1 Let $x \in \mathscr{M}_{0,n}$, then $\mathrm{Stab}(x)$ is isomorphic to a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$.

Therefore, we need to consider the types of finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$.

Lemma 2.1 Let G be a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$, and let $\overline{A_1}, \overline{A_2} \in G \setminus \{\overline{I_2}\}$. Let A_1 have characteristic subspaces V_{λ_1}, V_{λ_2} belonging to eigenvalues λ_1, λ_2. Then:

(i) $\overline{A_1A_2} = \overline{A_2A_1}$ if and only if $A_2V_{\lambda_i} = V_{\lambda_i}$ for $i=1$, 2 or $o(\overline{A_1}) = o(\overline{A_2}) = 2, A_2V_{\lambda_i} = V_{\lambda_j}$ for $1 \leqslant i \neq j \leqslant 2$.

(ii) Let $\overline{A_3} \in G$ and $o(\overline{A_1}) > 2$. If $\overline{A_2}$ and $\overline{A_3}$ commute with $\overline{A_1}$, then $\overline{A_2}$ commutes with $\overline{A_3}$.

(iii) If $\overline{A_1A_2} = \overline{A_2A_1}$ and $o(\overline{A_1}), o(\overline{A_2})$ are not all 2, then $\overline{A_1}, \overline{A_2} \in \langle \overline{A_1A_2} \rangle$.

(iv) Let $\overline{A_2A_1A_2^{-1}}$ commutes with $\overline{A_1}$ and $o(\overline{A_1}) > 2$. Then $\overline{A_2A_1A_2^{-1}} = \overline{A_1}^{\pm 1}$, and $\overline{A_2A_1A_2^{-1}} = \overline{A_1}^{-1}$ if and only if $o(\overline{A_2}) = 2$ and $A_2V_{\lambda_i} = V_{\lambda_j}$ for $1 \leqslant i \neq j \leqslant 2$.

Proof Since G is a finite group and $\overline{A_1}, \overline{A_2} \in G \setminus \{\overline{I_2}\}$, it follows that A_1 and A_2 are diagonalizable.

(i) Suppose that $\overline{A_2A_1A_2^{-1}} = \overline{A_1}$. Let $A_2A_1A_2^{-1} =$

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λA_1, then $\{\lambda_1,\lambda_2\}=\{\lambda\lambda_1,\lambda\lambda_2\}$. If $\lambda_1=\lambda\lambda_1$, then $A_2V_{\lambda_i}=V_{\lambda_i}$ for $i=1,2$. If $\lambda_1=\lambda\lambda_2$, then $\lambda_1=-\lambda_2$ and $A_2V_{\lambda_i}=V_{\lambda_j}$ for $1\leqslant i\neq j\leqslant 2$, then $o(\overline{A_2})=2=o(\overline{A_1})$.

Conversely, if $A_2V_{\lambda_i}=V_{\lambda_i}$ for $i=1,2$, then $A_2A_1=A_1A_2$, so $\overline{A_1A_2}=\overline{A_2A_1}$. If $o(\overline{A_1})=o(\overline{A_2})=2$ and $A_2V_{\lambda_i}=V_{\lambda_j}$ for $1\leqslant i\neq j\leqslant 2$, then $A_2A_1A_2^{-1}=\lambda_1\lambda_2A_1^{-1}$, so $\overline{A_2A_1A_2^{-1}}=\overline{A_1^{-1}}=\overline{A_1}$.

(ii) The proof of (ii) is trivial by (i).

(iii) We get $A_2V_{\lambda_i}=V_{\lambda_i}$ for $i=1,2$ by (i). Then there exists a matrix $P\in M_2(\mathbb{C})$, such that

$$A_1=P^{-1}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}P,\ A_2=P^{-1}\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix}P,$$

$$A_1A_2=P^{-1}\begin{pmatrix}\lambda_1\mu_1 & 0\\ 0 & \lambda_2\mu_2\end{pmatrix}P.$$

Since $\overline{A_1A_2}=\overline{A_2A_1}$, then $o(\overline{A_1})\mid o(\overline{A_1A_2})$ and $o(\overline{A_2})\mid o(\overline{A_1A_2})$, so there are $a,b\in\mathbb{Z}$ such that

$$\left(\frac{\lambda_1\mu_1}{\lambda_2\mu_2}\right)^a=\frac{\lambda_1}{\lambda_2},\ \left(\frac{\lambda_1\mu_1}{\lambda_2\mu_2}\right)^b=\frac{\mu_1}{\mu_2}.$$

Then $(\overline{A_1A_2})^a=\overline{A_1}$ and $(\overline{A_1A_2})^b=\overline{A_2}$, so $\overline{A_1},\overline{A_2}\in\langle\overline{A_1A_2}\rangle$.

(iv) We get $A_2A_1A_2^{-1}V_{\lambda_i}=V_{\lambda_i}$ for $i=1,2$ by (i). Note that $A_2V_{\lambda_1}$ and $A_2V_{\lambda_2}$ are the characteristic subspaces of $A_2A_1A_2^{-1}$. If $A_2V_{\lambda_i}=V_{\lambda_i}$ for $i=1,2$, then $\overline{A_2A_1A_2^{-1}}=\overline{A_1}$. If $A_2V_{\lambda_i}=V_{\lambda_j}$ for $1\leqslant i\neq j\leqslant 2$, then we can get $\overline{A_2A_1A_2^{-1}}=\overline{A_1^{-1}}$ and $o(\overline{A_2})=2$ by the proof similar to (i).

Lemma 2.2[2]

(ⅰ) The dihedral group of order $2n$ has a presentation $D_{2n}=\langle a,b\mid a^n=b^2=(ab)^2=1\rangle$.

(ⅱ) The alternating group A_4 has a presentation $A_4=\langle a,b\mid a^3=b^3=(ab)^2=1\rangle$.

(ⅲ) The symmetric group S_4 has a presentation $S_4=\langle a,b\mid a^4=b^2=(ab)^3=1\rangle$.

(ⅳ) The alternating group A_5 has a presentation $A_5=\langle a,b\mid a^5=b^2=(ab)^3=1\rangle$.

(ⅴ) The symmetric group S_5 has a presentation $S_5=\langle a_1,a_2,a_3,a_4\mid a_i^2=(a_ia_{i+1})^3=(a_ia_j)^2=1,1\leqslant i,j\leqslant 4,i+1<j\rangle$.

Lemma 2.3 Let G be a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$, and let P be a p-subgroup of G. Then:

(ⅰ) If $p>2$, then P is cyclic.

(ⅱ) If $p=2$, then P is a cyclic group or dihedral group.

Proof (i) Let $|P|>1$, so $|Z(P)|>1$, then P is abelian by Lemma 2.1(ii). Let $g\in P$ such that $o(g)=\max\{o(g')\mid\forall g'\in P\}$, then for any $g'\in P$, we get $g,g'\in\langle gg'\rangle=\langle g\rangle$ by Lemma 2.1(iii). Hence P is cyclic.

(ii) Let $|P|>2$, so $|Z(P)|>1$. If there is $g\in Z(P)$ such that $o(g)>2$, then P is cyclic by the same proof as (i). Now, we suppose that there is $\overline{A}\in Z(P)$ such that $o(\overline{A})=2$. Let $V_{\lambda_1},V_{\lambda_2}$ be the characteristic subspaces of A belonging to eigenvalues λ_1,λ_2. Let

$$\mathscr{A}=\{\overline{B}\in P\mid BV_{\lambda_i}=V_{\lambda_i},i=1,2\},$$

and let

$$\mathscr{B}=\{\overline{B}\in P\mid o(\overline{B})=2,BV_{\lambda_i}=V_{\lambda_j},1\leqslant i\neq j\leqslant 2\}.$$

So we have $P=\mathscr{A}\cup\mathscr{B}$ by Lemma 2.1(i). Let $g\in\mathscr{A}$ such that $o(g)=\max\{o(g')\mid\forall g'\in\mathscr{A}\}$. Note that $\{g'\in\mathscr{A}\mid o(g')=2\}=\{\overline{A}\}$. For any $a\in P$, we have $a,g\in\langle ag\rangle=\langle g\rangle$ by Lemma 2.1(i) and (iii), then $\mathscr{A}=\langle g\rangle$. If $\mathscr{B}=\varnothing$, then $P=\mathscr{A}$ is cyclic. If $b\in\mathscr{B}\neq\varnothing$, then $bg\in\mathscr{B}$ and $o(bg)=2$. For any $c\in\mathscr{B}\backslash\{b\}$, then $bc\in\mathscr{A}=\langle g\rangle$ by the definition of $\mathscr{B}$. Hence

$$P=\langle b,g\mid g^{o(g)}=b^2=(bg)^2=1\rangle$$

is a dihedral group.

Lemma 2.4[2] Let all Sylow subgroups of a finite group G be cyclic groups. If G is commutative, then G is a cyclic group; if G is not commutative, then G is a metacyclic group determined by the following definition relationship:

$$G=\langle a,b\rangle,a^m=b^n=1,b^{-1}ab=a^r,$$
$$\gcd((r-1)n,m)=1,\ r^n\equiv 1(\mathrm{mod}\ m),\ |G|=nm.$$

Lemma 2.5[3] Let G be a finite group, and let $O(G)$ be the largest normal subgroup of odd order in G. If G has dihedral Sylow 2-subgroups, then $G/O(G)$ is isomorphic to either

(ⅰ) a subgroup of $\mathrm{Aut}(\mathrm{PSL}(2,p^n))$ containing $\mathrm{PSL}(2,p^n)$, where $\mathrm{Aut}(\mathrm{PSL}(2,p^n))$ is isomorphic to the semidirect product of $\mathrm{PGL}(2,p^n)$ by a cyclic group of order n and p is an odd prime,

(ⅱ) the alternating group A_7, or

(ⅲ) a Sylow 2-subgroup of G.

Lemma 2.6[3] Let $G=\mathrm{PSL}(2,q)$ with $q=p^r$, where $q>3$ and p is an odd prime. If P is a Sylow p-subgroup of G, then $N_G(P)$ is a Frobenius group with a cyclic complement of order $\frac{q-1}{2}$ which acts irreducibly on P.

Theorem 2.2 Let G be a finite group. Then G is isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$ if and only if G is isomorphic to a cyclic group, a dihedral group, A_4, S_4 or A_5.

Proof Suppose that G is isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$. Let M be the largest normal subgroup of odd order in G. If $|M|>1$, we suppose M is not cyclic. By Lemma 2.3 and Lemma 2.4, we have

$$M=\langle a,b\rangle,a^d=b^k=1,b^{-1}ab=a^r,$$
$$\gcd((r-1)k,d)=1,\ r^k\equiv 1(\mathrm{mod}\ d),\ |M|=kd.$$

Since $b^{-1}ab=a^r\neq a$ and $d>2$, then $k=2$ by Lemma 2.1(iv), contradicting the hypothesis that M is a group of odd order. Hence M is a cyclic group. By suitable modification to proof of Lemma 2.3(ii), we can show that G is a cyclic group or dihedral group. If G is a 2-group, then G is a dihedral group by Lemma 2.3(ii). If G is not a 2-group and $|M|=1$, then G is isomorphic to a subgroup of Aut(PSL(2, p^r)) containing PSL(2, p^r) by Lemma 2.3 and Lemma 2.5, where p is an odd prime. If $p=3$ and $r=1$, then PSL(2, 3) is isomorphic to A_4 and Aut(PSL(2, 3)) is isomorphic to S_4, then G is isomorphic to A_4 or S_4. If $p^r>3$, then $N_G(P)$ is a Frobenius group with cyclic complement of order $\frac{p^r-1}{2}$ by Lemma 2.6, where P is a Sylow p-subgroup of G. Note that $N_G(P)$ is a dihedral group. Then $\frac{p^r-1}{2}=2$, then $p=5$ and $r=1$. Hence G is isomorphic to A_5 or S_5. Suppose that G is isomorphic to S_5. By Lemma 2.2, there is $g\in \mathrm{PGL}_2(\mathbb{C})$ such that $g^{-1}Gg=\langle \overline{A_1}, \overline{A_2}, \overline{A_3}, \overline{A_4}\rangle$, where

$$A_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, o(\overline{A_i})=o(\overline{A_iA_j})=2,$$

$$o(\overline{A_iA_{i+1}})=3, 1\leqslant i,j\leqslant 4, i+1<j.$$

However, by solving the equations, it is found that there are no $\overline{A_2}$, $\overline{A_3}$, $\overline{A_4}\in \mathrm{PGL}_2(\mathbb{C})$, which make the above formula hold. This leads to a contradiction. Hence G is not isomorphic to S_5.

Conversely, let G be isomorphic to a cyclic group, a dihedral group, A_4, S_4 or A_5. According to Lemma 2.2, we use the same method mentioned above and solve the equations, then we obtain that there is a subgroup G' in $\mathrm{PGL}_2(\mathbb{C})$ such that G' is isomorphic to G.

3 Types of stable subgroups

For $x\in\mathscr{M}_{0,n}$, we further discuss Stab(x) in this section for a more accurate description.

Lemma 3.1 Let $x\in\mathscr{M}_{0,n}$, and let $\sigma\in \mathrm{Stab}(x)\setminus\{(1)\}$. Then:

(i) A complete factorization of σ has r_1 1-cycles and r_2 d-cycles, where $0\leqslant r_1\leqslant 2$ and $r_1+r_2d=n$.

(ii) If there is $\tau\in \mathrm{Stab}(x)$ such that $\sigma\neq\tau$ and $\tau\sigma\tau^{-1}=\sigma^{-1}$. Then $|\{i|\sigma(i)=i\}|\neq 1$.

Proof Without loss of generality, we may assume $\sigma(2)\neq 2$. Let $x=[x_1,\cdots,x_n]$ such that $(x_1,x_2,x_3)=(0,\infty,1)$, and let $\sigma\cdot(x_1,\cdots,x_n)=\overline{A}\cdot(x_1,\cdots,x_n)$.

(i) Let $\sigma=\sigma_1\cdots\sigma_s$ be a complete factorization into disjoint cycles, and let this complete factorization of σ have r_i d_i-cycles, where $1\leqslant i\leqslant t$ and $1=d_1<d_2<\cdots<d_t$. Without loss of generality, we may assume $\sigma_1=(i_1\cdots i_{d_2})$. Since $\overline{A}\cdot(x_{i_1},x_{i_2})=(x_{i_2},x_{\sigma(i_2)})$ and $\overline{A^{d_2}}\cdot(x_{i_1},x_{i_2})=(x_{i_1},x_{i_2})$, it follows that there are linearly independent vectors α and β, such that they are eigenvectors of A^{d_2} but not A, so $\sigma^{d_2}=(1)$, then $r_3=\cdots=r_t=0$. For any $i\in\{i|\sigma(i)=i\}$, we have the vector $(x_i\ 1)^{\mathrm{T}}$ a eigenvector of A, so $0\leqslant r_1\leqslant 2$.

(ii) We need only consider the case $\{i|\sigma(i)=i\}\neq\varnothing$. Then we can take $i_0\in\{i|\sigma(i)=i\}$. Let $\tau\cdot(x_1,\cdots,x_n)=\overline{B}\cdot(x_1,\cdots,x_n)$. Since $\sigma\neq\tau$ and $\tau\sigma\tau^{-1}=\sigma^{-1}$, it follows that linearly independent vectors $B(x_{i_0}\ 1)^{\mathrm{T}}$ and $(x_{i_0}\ 1)^{\mathrm{T}}$ are eigenvectors of A. Then $\overline{B}\cdot x_{i_0}=x_{j_0}\neq x_{i_0}$, then $i_0\neq j_0\in\{i|\sigma(i)=i\}$, so $|\{i|\sigma(i)=i\}|=2$.

Proposition 3.1 Let $x\in\mathscr{M}_{0,n}$. Then one of the following holds:

(i) Stab(x) is a cyclic group of order m, where $m|n$ or $m|n-1$ or $m|n-2$.

(ii) Stab(x) is a dihedral group of order $2m$, where $m|n$ or $m|n-2$.

(iii) Stab(x) is isomorphic to A_4 or S_4.

(iv) Stab(x) is isomorphic to A_5.

Proof The proof is trivial by Theorem 2.1 and Lemma 3.1.

From Proposition 3.1, we can get the possible types of stable subgroups. Next, we do not fix n, and then prove the existence of these possible types by some examples.

Lemma 3.2 Let $x\in\mathscr{M}_{0,n}$. Then:

(i) If Stab(x) is isomorphic to A_4, then $n\equiv 0,4,6,8 \pmod{12}$.

(ii) If Stab(x) is isomorphic to S_4, then $n\equiv 0,6,8,12 \pmod{24}$.

(iii) If Stab(x) is isomorphic to A_5, then $n\equiv 0,12,20,30 \pmod{60}$.

Proof Consider Stab(x) acts on the set $\{1,2,\cdots,n\}$. For any $1\leqslant i\leqslant n$, the stabilizer of i, denoted by G_i, does not contain dihedral groups by Lemma 3.1, then the size of the orbit of i is $|\mathrm{Stab}(x)|/|G_i|$, where G_i is cyclic. Therefore, the conclusion is obtained by Lemma 3.1.

Example 3.1 The following examples show that each finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$ happens as a stable subgroup.

(i) Let $x=[1,\zeta_3,\zeta_3^2,0]\in\mathscr{M}_{0,4}$. Clearly (1 2 3), (1 2)(3 4), (1 3)(2 4), (1 4)(2 3) $\in$ Stab(x), then Stab(x)$=A_4$.

(ii) Let $x=[1,\zeta_4,\zeta_4^2,\zeta_4^3,0,\infty]\in\mathscr{M}_{0,6}$. Obviously (1 2 3 4) $\in$ Stab(x). Since

$$\overline{\begin{pmatrix}1&-1\\-1&-1\end{pmatrix}}\cdot(1,\zeta_4,\zeta_4^2,\zeta_4^3,0,\infty)=(0,\zeta_4^3,\infty,\zeta_4,1,\zeta_4^2),$$

it follows that (1 5)(2 4)(3 6) $\in$ Stab(x). According to Proposition 3.1, it follows that Stab(x) is isomorphic to S_4.

(iii) Let $x=[1,\zeta_5,\zeta_5^2,\zeta_5^3,\zeta_5^4,\frac{1+\zeta_5^2}{1+\zeta_5},\frac{\zeta_5+\zeta_5^3}{1+\zeta_5},\frac{\zeta_5^2+\zeta_5^4}{1+\zeta_5},\frac{\zeta_5^3+\zeta_5^5}{1+\zeta_5},\frac{\zeta_5^4+\zeta_5^6}{1+\zeta_5},0,\infty]\in\mathscr{M}_{0,12}$. Obviously (1 2 3 4 5)(6 7 8 9 10) $\in \mathrm{Stab}(x)$. Since

$$\overline{\begin{pmatrix}1 & 1-\zeta_5-\zeta_5^4\\ 1 & -1\end{pmatrix}}\cdot (1,\zeta_5,\zeta_5^2,\zeta_5^3,\zeta_5^4,\frac{1+\zeta_5^2}{1+\zeta_5},\frac{\zeta_5+\zeta_5^3}{1+\zeta_5},\frac{\zeta_5^2+\zeta_5^4}{1+\zeta_5},\frac{\zeta_5^3+\zeta_5^5}{1+\zeta_5},\frac{\zeta_5^4+\zeta_5^6}{1+\zeta_5},0,\infty)=(\infty,\zeta_5^4,\frac{\zeta_5^4+\zeta_5^6}{1+\zeta_5},\frac{1+\zeta_5^2}{1+\zeta_5},\zeta_5,\zeta_5^3,\frac{\zeta_5^3+\zeta_5^5}{1+\zeta_5},0,\frac{\zeta_5+\zeta_5^3}{1+\zeta_5},\zeta_5^2,\frac{\zeta_5^2+\zeta_5^4}{1+\zeta_5},1),$$

it follows that

$$(1\ 12)(2\ 5)(3\ 10)(4\ 6)(7\ 9)(8\ 11)\in \mathrm{Stab}(x).$$

According to Proposition 3.1, it follows that $\mathrm{Stab}(x)$ is isomorphic to A_5.

(iv) Let $n\geqslant 4$, $x=[1,\zeta_n,\zeta_n^2,\cdots,\zeta_n^{n-1}]\in\mathscr{M}_{0,n}$. Clearly $\sigma=(1\ 2\ \cdots\ n)\in\mathrm{Stab}(x)$. Since

$$\overline{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}\cdot(1,\zeta_n,\zeta_n^2,\cdots,\zeta_n^{n-1})=(1,\zeta_n^{n-1},\zeta_n^{n-2},\cdots,\zeta_n),$$

it follows that

$$\tau=(2\ n)(3\ n-1)\cdots(d\ n+2-d)\in\mathrm{Stab}(x),$$

where d is integral part of $\frac{n+2}{2}$, then $o(\tau)=o(\tau\sigma)=2$. If $n\geqslant 6$, then $o(\sigma)=n>5$. If $n=4$ or 5, then $\mathrm{Stab}(x)$ is not isomorphic to S_4 or A_5 by Lemma 3.2. So $\mathrm{Stab}(x)$ is isomorphic to D_{2n} by Proposition 3.1.

(v) Let $x=[1,-1,0,\infty]\in\mathscr{M}_{0,4}$. Then $\mathrm{Stab}(x)=\{(1),(1\ 2)(3\ 4),(1\ 3)(2\ 4),(1\ 4)(2\ 3)\}\cong D_4$.

(vi) Let $x=[1,\zeta_3,\zeta_3^2,0,\infty]\in\mathscr{M}_{0,5}$. Clearly (1 2 3), (2 3)(4 5) $\in\mathrm{Stab}(x)$. Then $\mathrm{Stab}(x)$ is isomorphic to D_6 by Lemma 3.2 and Proposition 3.1.

(vii) Let $n\geqslant 2$, and let $x=[1,\zeta_n,\cdots,\zeta_n^{n-1},2,2\zeta_n,2,\cdots,2\zeta_n^{n-1},0]\in\mathscr{M}_{0,2n+1}$. Since

$$\overline{\begin{pmatrix}\zeta_n & 0\\ 0 & 1\end{pmatrix}}\cdot(1,\zeta_n,\cdots,\zeta_n^{n-1},2,2\zeta_n,2,\cdots,2\zeta_n^{n-1},0)=(\zeta_n,\cdots,\zeta_n^{n-1},1,2\zeta_n,2,\cdots,2\zeta_n^{n-1},2,0),$$

it follows that

$$\sigma=(1\ 2\ \cdots\ n)(n+1\ n+2\ \cdots\ 2n)\in\mathrm{Stab}(x).$$

Then $\mathrm{Stab}(x)$ is isomorphic to a cyclic group or A_4 by Lemma 3.1(ii) and Proposition 3.1. According to Lemma 3.2, it follows that $\mathrm{Stab}(x)$ is not isomorphic to A_4, then $\mathrm{Stab}(x)$ is cyclic. Since

$$\overline{\begin{pmatrix}\zeta_{2n} & 0\\ 0 & 1\end{pmatrix}}\cdot 1\notin\{1,\zeta_n,\cdots,\zeta_n^{n-1},2,2\zeta_n,2,\cdots,2\zeta_n^{n-1},0\},$$

it follows that $\mathrm{Stab}(x)$ is a cyclic group of order n.

(viii) Let $x=[x_1,x_2,\cdots,x_n]\in\mathscr{M}_{0,n}$, $n\geqslant 5$, where $x_i\in\mathbb{C}$ for $1\leqslant i\leqslant n$ and x_n is transcendental over $\mathbb{Q}(x_1,\cdots,x_{n-1})$. Suppose there is $(1)\neq\sigma\in\mathrm{Stab}(x)$. Then there is $\bar{A}\in\backslash GL_2(\mathbb{C})\backslash\{\bar{I_2}\}$ such that

$$\bar{A}\cdot(x_1,x_2,\cdots,x_n)=(x_{\sigma(1)},x_{\sigma(2)},\cdots,x_{\sigma(n)}).$$

Without loss of generality, we may assume $A=\begin{pmatrix}a & b\\ 1 & c\end{pmatrix}$. We can take three different numbers i_1,i_2,i_3 in the set $\{1,\cdots,n-1\}$, such that $\sigma^{-1}(n)\notin\{i_1,i_2,i_3\}$. Then

$$\frac{ax_{i_1}+b}{x_{i_1}+c}=x_{\sigma(i_1)},\ \frac{ax_{i_2}+b}{x_{i_2}+c}=x_{\sigma(i_2)},\ \frac{ax_{i_3}+b}{x_{i_3}+c}=x_{\sigma(i_3)}.$$

So $a,b,c\in\mathbb{Q}(x_1,\cdots,x_{n-1})$. But $\frac{ax_{\sigma^{-1}(n)}+b}{x_{\sigma^{-1}(n)}+c}=x_n$, then x_n is algebraic over $\mathbb{Q}(x_1,\cdots,x_{n-1})$. This leads to a contradiction. Hence $\mathrm{Stab}(x)=\{(1)\}$.

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Conflict of interest

The author declares no conflict of interest.

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$\mathscr{M}_{\mathbf{0},n}$ 上的 S_n 作用的稳定子群

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摘要: 考虑对称群 S_n 在 $M_{0,n}$ 上的作用,得到了所有可能的稳定子群.

关键词: 模空间;对称群;群作用;稳定子群